Some Neglected Niches in the Understanding and Teaching of Numbers and Number Systems

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Abstract: Cultural questions have attended arithmetic since it began to develop in ancient times. They include possible differences between integers and non-integral numbers and in operating with them, religious and mystical uses and interpretations, the roles of zero, extensions to infinite numbers, and representing numbers by numerals in ways which aid calculation (including the use of algebra). The selection of historical examples given here concentrates on aspects of numbers which are not well known but which could be used in teaching, either at school or undergraduate level. Comments on educational utility are given, mostly at the end of each section; the word "student" refers to learners of all ages.

Kurzreferat: Seit ihrer Entwicklung in der Antike wurde die Arithmetik von kulturellen Fragen begleitet. Sie bezogen sich auf mögliche Unterschiede zwischen ganzen und nicht-ganzen Zahlen und Operationen mit diesen, religiöse und mystische Gebräuche und Interpretationen, die Rolle der Null, Erweiterungen auf unendliche Zahlen sowie auf die Darstellung von Zahlen durch Zahlsymbole, die das Rechnen erleichtern (inkl. der Algebra). Die hier getroffene Auswahl historischer Beispiele umfaßt Aspekte von Zahlen, die weniger bekannt sind, aber im Unterricht an Schulen und Hochschulen bis zum ersten Abschluß behandelt werden können. Didaktische Hinweise werden, meist am Ende eines Kapitels, gegeben. Der Begriff "student" bezieht sich auf Lernende aller Altersstufen.

ZDM-Classification: A30, F10

1. Ancient number things

The origins of numbers and their arithmetic cannot be known for certain, but it is likely that both arose in all cultures from counting and commercial exchange. In such contexts numbers can be regarded philosophically merely as marking stages in counting. Ancient texts, especially Greek and Chinese, give the impression that (say) 3 is about three *things* of some kind, even if no particular kind is mentioned in general explanations. The word "calculus" in Latin means "pebble", hinting at the role that tokens had played in the development of counting, and maybe even of writing itself (Schmandt-Besserat 1992).

Fractions and ratios raise further conceptual difficulties; for a rational number seems to be a different sort of thing from an integer, even though the rules for adding and subtracting are similar (Benoit and others 1992). And irrational numbers are even more "risky" ("lawless" would be a better translation for the Greek word "alogos" than "irrational").

In applications to abstract concepts (in science, for example), the perplexities can become great. Regarding multiplication and division, a long tradition stemmed from the Greeks of working with ratios and proportions (that is, sentences relating a pair of ratios) rather than equations. Let a, b, c, and d be numbers, and contrast the proportion

$$a : b :: c : d \text{ with } ad = bc \text{ and } a : c :: b : d. \qquad (1)$$

If a and b refer to the same kind of thing (force, say, or volume), and so do c and d (but *not* necessarily the same kind), then $a : b$ and $c : d$ will be *dimensionless* ratios, so that their (in)equality with each other can(not) be asserted in $(1)_1$ via "::" without qualms. This was the way Euclid worked with ratios and proportions, relating, say, a pair of lines to another pair of lines or to a pair of numbers (Grattan-Guinness 1996). However, $(1)_2$ poses a conceptual difficulty about dimensions; ad and bc may be equal, but in what units? Similarly, the brother proportion $(1)_3$ was maybe better avoided if the ratios were of two different kinds of thing.

Such emphases on proportions may have been transferred to metaphysical views about the construction of the heavens, the chronology of myths, and other cultural concerns in which rhythm or periodicity was held to apply (McClain 1978). The phrase "music of the spheres" may carry more overtones (as it were) than is now realised.

In several cultures an early interest in proportion arose from musical harmony. The octave was set at 2:1, the perfect fifth at 3:2, and so on – but how can they fit together, since no power of 2 can equal any power of 3/2? Various systems of "temperament" have been devised to place the seven tones or 12 semitones within the octave, especially in the late Middle Ages (Cohen 1984); for example, the devilish character of the mid-note "tritone" (the augmented fourth) was associated with the irrational ratio $\sqrt{2} : 1$.

During that period the theory of ratios became rather more arithmetical and less linked to geometry via dimensions. In the new approach it was permitted to state, say,

$$a : b = c \text{ and to deduce that } a = b \times c. \qquad (2)$$

The distinction between a fraction and a ratio tended to disappear, and proportions were replaced by equations (Sylla 1984). Indeed, for a long time the words “ratio” and “proportion” have been treated as synonyms, a sloppiness that was not practised in earlier centuries.

The use of ratios was not confined to ancient or older times. For example, mechanics through the 18th century and 19th century mathematical physics used in order ratios to avoid exactly the same kinds of questions about force, say, or electrostatic charge.

Comment

Ratios deserve a much greater role in teaching than they normally receive; for their long history shows them to be a very natural way for human beings to compare. The well-known difficulties in teaching fractions can be alleviated by converting to ratios (so that 4/7 becomes 4:7) and pointing to examples in the classroom and ordinary life where they occur; music is a particularly good source for the latter context, or even both of them.

2. Integers with properties

The autonomy of integers and their arithmetic from empirical factors became more evident when they began to be regarded as objects in some (usually unspecified) sense; that is, they possessed *properties* such as being prime, or factors of other integers. Diophantos (flourished around 250 AD) used such properties to find solutions to linear equations, many of which involve rational as well as integral solutions.

But the importance of proportions just described is an example of cultural contexts in which integers were treated as objects. They involve forms of numerology, in which a community granted special status to certain integers by associating them with their metaphysical or religious beliefs. Probably they saw integers as invariants, which have always been a strong theme in mathematics; things which remain the same while other things change around them. A closely related doctrine was gematria, created when a culture had developed a written alphabet; each letter was associated with some integer, and a word or phrase of the language took the integer given by the sum of those of its constituent letters.

Several Greek thinkers, especially Plato and followers such as Iamblichos (flourished in the 3rd century AD), advocated numerology. One important category was the “triangle numbers”, integers starting 3, 6, 10, ... which can be written as 1+2+3+...; the name indicates the property that they can be represented spatially by triangles of tokens. Maybe this property was one source of the 3ness of orthodox Christianity, in which Jesus is held to have lived for 3×10 years in obscurity before prosecuting a 3-year ministry and advocating the Trinity prior to dying in 3some on a cross but coming back to life 3 days later. After his resurrection Jesus appeared 3 times before the remaining apostles; the second time was before 7 (sic) of them on the Sea of Galilee, whereupon Peter catches fish to the total of 153 (John 21:11) – the triangle number of 17, which was the number of principal sects and societies in the Jewish Kingdom of the time.

The full account is given in the New Testament, which contains 27 books: 3^3, the Trinity propounded across the 3 dimensions of space. Such features embody profoundly meaningful metaphysics to cognisant Christians. There were many of them in the Middle Ages, deeply aware of both numerology and gematria (Hopper 1938); but now it is usually ignored or even derided. Thus believers do not know a significant factor in their belief system.

These examples show connections between mathematics and (a) religion, a rich aspect of mathematics which is often ignored but which is not even confined to arithmetic (mechanics and probability theory are among other branches). The Jewish tradition is very strong here. For example, the 22 letters of its alphabet are understood as “Alpeh” followed by 3 7s respectively of grace, mercy and strict justice, while 10 features as the number of connected circles in the Sefirot tree of the Kabbalah (Judaica 1971). For Hindus, 9 is especially important, as the number of parts of the religion itself and of the points in their special symbol, the swastika.

Comment

The arithmetic involved here is trivial; but its cultural weight is substantial, and gives a route into multi-ethnic education.

3. Algebra within and beyond arithmetic

The advocacy of algebra from the late Middle Ages onwards was helped by its prowess in finding the roots of polynomial equations; but it also served to generalise arithmetic by expressing the basic properties of any numbers, known or unknown real ones, and even complex. This aspect was also encouraged by the neo-Humanist movement of the late 16th century, which extolled the merits of the inheritance from Greek culture (Klein 1968). An important example is Rafael Bombelli (1526–1572), who was shown a manuscript by Diophantos and rewrote part of his *Algebra* before publishing it in 1572 to include solutions to Diophantine equations. As his bald one-word title shows starkly, he wished that now algebra was not merely a means to aid arithmetical calculation but a brother discipline in mathematics.

Another line emanating from Bombelli was continued fractions, an arithmetical analogue of the Euclidean algorithm. Let $b > a$, and divide and use the successive remainders $c, d, \ldots$ as follows:

$$b/a = C + c/a = C + 1/(D + d/a) = \\ C + 1/(D + 1/[E + e/a]) = \ldots . \qquad (3)$$

Some of the new integers $C, D, \ldots$ may be zero; if there is a remainder $b, c, \ldots$, then the algorithm stops, and b/a is a rational fraction. Properties of continued fractions have enriched both arithmetic and algebra, although the topic remains curiously fugitive; for example, it is rarely taught.

Bombelli’s contemporary François Viète (1543–1603) was also influenced by neo-Humanism. For he viewed the new art of algebra as “analytic” in the sense of the proof method advocated by Greeks such as Pappos, which starts out from the theorem and ended up with axioms or with previously known results, in contrast with the “synthetic” proof method, which proceeds from assumptions to theorem and was associated with geometry. This pair of associations endured in the philosophy of mathematics, although often to little benefit as geometry turned more analytical and algebra used synthetic proof methods in its more advanced theories. Further, the philosophy of algebra was still tied down by geometrical considerations; for example, since space had only three dimensions, then for Viète $xxxx$ was the "plano-planum" power of x, not its "quartum". Liberation from such constraints came only in the qualm-free "x^4" used by René Descartes (1596–1650) in his *Géométrie* (1637); however, as his title shows, algebra served as handmaiden to his creation of analytic geometry. (This was *not* co-ordinate geometry, by the way, where a system of axes is imposed upon a diagram; G. W. Leibniz (1646–1716) was a pioneer here, in the development of his differential and integral calculus from the 1670s.)

Attached to the development of algebra was the status of negative numbers, which have suffered a nervous press over the centuries. For some mathematicians (including Descartes) their status was linked to that of complex numbers, since both kinds of number arose in connection with solving polynomial equations. Although one can naturally think of interpretations of negatives (as financial debts, for example, or as numbers marked off in the direction opposite to that of the positive numbers), their legitimacy as self-standing objects has frequently been questioned. Consider the two equations

$$5 - 3.5 = 1.5 \text{ and } 3.5 - 5 = -1.5. \qquad (4)$$

Philosophically speaking, can the second equation stand on its own, or is it only a way of really saying the first? The second position was preferred by many mathematicians from the days of early algebra until the 19th century, especially in England. Rather greater confidence for negative numbers was evident among Continentals. Immanual Kant (1724–1804) argued in 1763 that negation should be construed as the dialectic opposite to positiveness (it is curious that one cannot say "position"!) rather than as expression of its absence. Later, the abbot Condillac (1714–1780) saw algebra as the language par excellence within his semiotics, and so treated negative quantities on a par with positive ones in his account of ordinary algebra, published posthumously in 1807 under another striking title: *La langue des calculs.*

Comment

This is a case where the historical lesson is negative; for the two natural interpretations are easy to teach (the latter as a staple in financial mathematics, which is becoming a trendy topic in education). The failure to draw upon such cases in the past reflects, I suspect, *the lack of model-theoretic thinking in mathematics in those times.* This view, which has flourished only in this century, allows negative numbers to be interpretable in such contexts while not in others. It is worth teaching explicitly, though the technicalities of model theory should be left until well into the undergraduate career.

4. Number systems and calculation

The Babylonians worked with a number system based on 60, which still leaves its traces on our system of time keeping. They also used 10 as their base, as have most cultures. A system of counting based on 12 is more convenient, since it can be done on one hand by counting the knuckles of the fingers with the thumb; but sadly this system never became popular, although it is richer in that 12 contains more factors than does 10. Note that both 10 and 12 are both factors of 60, which may have been a common source for both integers: against frequent statement, the linguistic evidence is *not* strong that 10 came from the numbers of fingers and thumbs or of toes; maybe its status as the triangle number of 4 also played some role.

A variety of other number systems have been known; for example, the Mayans took 1 and 5 as their basic units in a 20-place arithmetic (Lounsbury 1978). Not all non-standard systems are ancient; for modern science is still developing a library of adjectives (mostly derived from Greek) for these numbers 10^n:

$n =$ …	-18	-15	-12	-9	-6
name: …	atto-	femto-	pico-	nano-	micro-
	-3	3	6	9	12
	milli-	kilo-	mega-	giga-	tera-.

(5)

Some systems of numerals reflect the operations carried out. For instance, if in our Hindu-Arabic system (hereafter, "HA") I add 6 apples to 2 apples, then I obtain 8 apples; but in Roman numerals I go from ii and vi to viii, and the sign "viii" exhibits the process of adding ii and vi. This feature seems to have appealed to accountants and others in the 16th century as a means of controlling honesty in financial records; so they opposed the introduction of HA. However, HA has other advantages, and became dominant. Simon Stevin (1548–1620?) pointed out one benefit: if HA were extended to the use of decimal expansion of numbers, then magnitudes could be easily compared. Indeed, how else can one readily tell whether, say, 83/35 is less or greater than 71/29?

Although virtuosi on the abacus could be found until recently in Soviet supermarkets, gift at calculation has been most frequently found among the exponents of HA. Many interesting and indeed little-understood questions in psychology arise; for example, "lightning calculators" often do not know themselves how they calculate at such speed, and most of them have no particular gift for mathematics itself (Smith 1983).

At the other extreme come innumerate people, for whom arithmetic is a frightening subject. While social conditioning from bad educational experience is a major factor, mental incapacity also seems to be involved, especially when it occurs in some forms of calculation and not in others.

One feature which inhibits proficiency in multiplication and division in HA is that the standard methods do not make full use of the number system. To multiply two integers, the natural way should be to take the units of each integer to find the units place of the product; multiply the units of each integer with the tens of the other one to find the tens place; and so on. This insight, which is due to the Vedic tradition in India (Shankarachaya 1965), yields the product very easily: for example,

$$\begin{array}{lr} \text{First integer} & 4\,6\,7 \\ \text{Second integer} & \underline{5\,8} \\ \text{Product} & \underline{\underline{2\,7\,0\,8\,6}} \\ \text{Carrying} & 7\,8\,5 \end{array} \qquad (5)$$

The hundreds figure, 0, arises thus: $8\times4+6\times5+7\times0 = 62$; take the 8 carried from the tens to get 70; hence the 0, and carry 7 to the thousands; read the 70 diagonally in the last two lines. A companion method for doing (not very) long division was also known.

A major issue in the development of methods of calculation in the Middle Ages was *whether or not the details were retained, for purposes of checking*. The non-retentive methods, such as the use of tokens mentioned in §1, which advanced to moving pebbles to various positions on an abacus (Latin for "flat surface") to represent values, became known as "algorist"; the other methods were "abbacist". (Note the two b's now used to avoid the clash of names; the latter sense came from Fibonacci's "Liber Abbaci" (1202), an important source for Europe of Arabic and Hebrew arithmetic and algebra.) Gradually abbacist methods came to prevail, mainly because of the advantages of being able to check; Figure 1 shows a classic illustration of the triumph of an abbacist over her plodding algorist companion.

Fig. 1: *Happy abbacist, sad algorist (as in Greek drama?): propaganda in Georg Reisch's* Margarita philosophica, 1504. The algorist has these numbers on his board: on our left, 2+30+50=82; on our right, 1+40+200+1,000=1,241.

Comments

1) The issue of checking calculations bears upon modern education in a way that is often overlooked: namely, that *an electronic calculator is an algorist device*, in that no means are available to check the answers delivered. Abbacist principles should be emphasised in teaching, at least that the student should have some idea of the order of magnitude of the answer before pressing the buttons.
2) Errors in calculation can provide amusing but instructive cases. For example,

$$\not{6}4/1\not{6} = 4 \text{ can be generalised to } \not{a}b/1\not{a} = b, \qquad (6)$$

which is an indeterminate problem and so requires the student to examine options rather than proceed to the answer(s) in the usual robotic manner. A little work on factors in $(6)_2$ will furnish also

$\not{9}5/1\not{9} = 5$; and there are variants, such as

$$\not{9}8/4\not{9} = 8/4. \qquad (7)$$

5. The logic and set theory of arithmetic

The arithmetic of positive integers received two new levels of foundation in the 1880s. C. S. Peirce (1839–1914) and Giuseppe Peano (1858–1932) both put forward axiomatisations based on the notion of 1 as the initial integer and on the operation of successorship (2 as the successor of 1, and so on). One of the axioms was the principle of mathematical induction, in the form that if a property applies for $n = 1$, and if applicable for any value of n then is also for its successor, then it applies for all n. The formulation was simplified in 1907 by Mario Pieri (1860–1913). Richard Dedekind (1831–1916) gave a somewhat similar treatment in 1888, in terms of a notion of chain (a relation mapping a set of integers onto itself in such a way that all the successors of some initial integer were obtained); but he went further in proving a theorem which legitimated proofs by mathematical induction by providing a justification for inductive definitions. Van Heijenoort 1967 contains many original texts on foundations.

In 1884, Gottlob Frege (1848–1925) had gone a layer below this one when he gave nominal definitions of integers, which Bertrand Russell (1872–1970) was to find independently in 1901. They defined integers as sets of sets, starting with 0 and the unit set of the empty set $\emptyset$, 1 as the set of sets isomorphic (that is, in one-one correspondence) to $\{\emptyset\}$, and so on upwards but not in a vicious circle. The ascent was infinite for Russell, since he accepted the theory of transfinite numbers (which is described in the next section).

Frege proceeded to further definitions of real numbers (whether integral or not) via expansions of their non-integral part in the bicimal power-series

$$\sum_r a_r 2^{-r}, \text{ where each } a_r = 0 \text{ or } 1; \qquad (8)$$

Russell defined rational and irrational numbers as certain sets of integers and of rationals respectively. Arithmetical operations were defined in terms of the corresponding combinations of sets (for example, addition from the union of disjoint sets). Both men also had means of defining negative numbers.

Both men also defined sets in terms of propositional functions, for these procedures formed part of their logicist theses, that arithmetic (for Frege) or all "pure" mathematics (for Russell) could be derived solely from logical principles and procedures. A modern variant of their approach regards integers as quantifiers: the sentential form "there are ... apples here" is bound into a sentence by the insertion of the integer in question (Bostock 1974, 1979).

Comments

1) The vision behind these enterprises, especially Russell's, was the unification of arithmetic, mathematical analysis and geometries under the umbrella of set theory and mathematical logic. The purpose was epistemological, concerned with restructuring and "justifying" known theories by locating them within such foundations; there was no educational or heuristic content, although some parts can be taught in late student years at university. Sadly, half-understood versions percolated down to the educational community, granting set theory a central place in the "New Mathematics" of the 1960s. One factor was played by Jean Piaget's misunderstanding of Russell's enterprise, especially in giving such a grossly exaggerated place to isomorphisms in "the child's conception of number" (Piaget 1952).
2) The adjective "New" shows already the absence of historical knowledge (after all, Sibelius was hardly New Music at that time), never mind the original purpose and its limitations. The basic error in the approach was *premature rigour and generality*, solving problems which the student cannot have encountered in the first place (Grattan-Guinness 1973). Further, mistakes made in the texts testify to the subtlety of set theory, such as distinguishing properties of a set from those of its members. And the idea that the theory is the most general way of handling collections *is doubly mistaken*. Firstly, it assumes membership always to be well-defined (that is, true or false): fuzzy set theory takes care of the many exceptions (Dubois and Prade 1980). Secondly, it only allows members to belong once to the set – an elementary limitation, as evidenced by multiple roots of a polynomial equation: multiset theory is needed instead (Rado 1975).

6. Transfinite arithmetic

From 1883 Georg Cantor (1845–1918) also defined integers from sets, but by a very different means: from a given set M, abstract (that is, mentally ignore) the nature of its elements to leave behind the "order-type" $\overline{M}$ in which its members lie; then abstract the order to leave behind the cardinal number $\overline{\overline{M}}$ of M (Dauben 1979). The role which he gave to mental acts was rejected by most of his contemporaries, although some residue lies in the phenomenological interpretation of integers proposed by Edmund Husserl (1859–1928) from 1891, partly under influence from Cantor (Willard 1984).

Further, Cantor realised that infinities came in different sizes, and so introduced the concept of inequalities between infinitely large integers. He defined the sequence of transfinite ordinals, starting with the smallest of them, ω, assumed to exist after the finite ordinals, and with no predecessor ordinal:

$$1, 2, 3, \ldots\ \omega, \omega+1, \ldots\ 2\omega, 2\omega+1, \ldots \\ 3\omega, \ldots\ \omega^2, \ldots\ \omega^3, \ldots . \qquad (9)$$

He also found a way of dividing this literally infinite sequence into "number classes" whose sizes (that is, cardinalities independent of order) were shown to be different. The first one, comprising the finite integers, took the smallest such cardinal number, $\aleph_0$; the second class, starting at ω and defined by the property that the count up to any ordinal member could be reordered into a set of size $\aleph_0$, was itself of the next larger cardinality, $\aleph_1$, and so on, for infinitely ever.

Another of Cantor's innovations was to realise that the members of a set may be ordered in different ways. In order to preserve the generality of arithmetic, he asserted the "well-ordering principle", that any set could have its elements arrayed in the order exemplified by (9). This was to be one of his unsolved problems, of which the proof (in 1904) turned out to require the axiom of choice (Moore 1982). The main advocate of this axiom, which caused much controversy among mathematicians and philosophers for its various forms and especially for its non-constructive character, was Ernst Zermelo (1871–1953); but Russell found a form slightly earlier, in the context of defining the infinite product of numbers.

Cantor developed an arithmetic for each kind of integer, different from each other and from that of finite arithmetic; for example,

$$\omega + 1 > \omega, \quad 1 + \omega = \omega; \text{ and } \aleph_0 + \aleph_1 = \aleph_1. \qquad (10)$$

However, his sequence (10) also led to paradoxes of the supposedly greatest ordinal, and also the greatest cardinal; for either such number N, both $N = N$ and $N > N$. Avoiding these paradoxes, and those like Russell's concerning sets alone, led to modified definitions of numbers as sets of sets. In particular, the type theory in Russell's post-paradox logicism required that integers be defined and their arithmetic be developed for each type.

Comments

This material forms a good undergraduate course. I find that the axioms of choice are rather more fruitful source of appreciation than the details of transfinite arithmetic, partly because the range of consequences for mathematics far exceeds set theory and arithmetic, and partly because the wide range of equivalent axioms expose fine examples of apparent (non-)constructivity in mathematics.

7. Much ado about zero

The Babylonians usually indicated zero by an empty space; but then it is hard to distinguish, for example, 3 5 from 3 5 (in HA, 305 from 30,005). So a sign was needed; they had one, looking something like ⋞; but it seems to have been used only as place marker for the blanks. The great step of using it *also* as a number which could be combined arithmetically with other numbers seems to be of Indian origin.

Culturally, the status of zero, its mis-identification with nothing, has been widespread concern (Rotman 1987).

An important example is the playwright who wrote as "William Shakespeare", especially in his "King Lear" where nothing is everything. He was working at the turn of the 16th and 17th centuries, precisely when HA was coming into general use in Britain; doubtless this process heightened his awareness.

As for signs, HA has "0"; its origins are not known for certain, for it may lie in Greek, Indian and Chinese mathematics (possibly independently), perhaps from the 2nd century AD onwards. In addition, "o" is the first letter of the word "ouden" in Greek for nothing. It may also have been proposed as a sign for the vagina, as the nothing out of which things are born; for it is well known that fertility and sexual symbols played a prominent role in cultural and religious life of ancient civilisations. Other symbols used were similar to "0"; a dot •, and in Greek sometimes either "ϕ" or "θ".

Even when the need for a sign was recognised, much philosophical perplexity surrounded zero, especially for its failure to satisfy the cancellation law ($a\times 0 = b\times 0$ without $a = b$). In addition, it was usually associated (too) closely with nothing. Even in their formulations of arithmetic reported in §5 and §6, Cantor and Dedekind both always began with 1: Cantor could not define 0, for he would have had to abstract from the empty set $\emptyset$ (which makes surprising his notation "$\aleph_0$"!) Only Russell and Frege clearly understood the difference between nothing, $\emptyset$, and 0. Since then the distinction has become firmly established, even in systems where a mathematician may *choose* to conflate them. For example, the system of ordinals proposed in 1923 by Johann von Neumann (1905–1958) goes as follows:

$$0 := \emptyset, \quad 1 := \{\emptyset\}, \quad 2 := \{0, 1\}, \ldots, \\ \omega := \{0, 1, \ldots\}, \ldots . \qquad (11)$$

Comments

1) The teaching of zero is often deplorable, especially when it is stated to be nothing. This is quite incorrect; for zero has properties, such as $7+0 = 7$ (7+ nothing is not defined).
2) One can also use zero to emphasise the distinction between equality and identity; in addition to standard cases such as

   $$2 + 2 = 4, \text{ examples such as } 0 + 0 = 0 \qquad (12)$$

   show the difference still more starkly (two zeros on the left hand side, only one on the right, hence not identical). Identity is a difficult philosophical concept with which to work; for educational purposes in mathematics, it is best presented in terms of identification (of 4 as the sum of 2 and 2, say).

8. Formalisms and incompleteness

The word "formalism" occurs regularly in connection with the philosophy of arithmetic, but various types should be distinguished. There is a type of formalism in Cantor, in that he saw that the consistent construction of transfinite numbers guaranteed their existence. He had no metatheory in which consistency could be proved, and this lack may have been one of stimuli for David Hilbert (1862–1943) to devise metamathematics as a theory in which it (and completeness) of an axiomatised theory could be studied.

Hilbert's formalist programme, which flourished mainly in the 1920s and 1930s after launch at the beginning of the century (Detlefsen 1986), took arithmetic in a form broadly similar to Peano's axiomatisation mentioned in §5, and viewed its foundation as sufficient to provide the foundations of much (maybe "all") mathematics. But his hope of demonstrating the consistency of arithmetic was set back by a theorem proved in 1931 by Kurt Gödel (1906–1978); for its corollary shows that consistency can be proved only in a meta-theory richer than arithmetic itself, contrary to Hilbert's vision of a more primitive meta-theory, then still more primitive meta-meta-theory, ... Further, Gödel's main theorem refuted Russell's (intuitive) belief that his logicist axiomatisation of arithmetic would be complete.

In addition, Gödel's proof-method based upon expressed the meta-theory in arithmetical terms was a principal stimulus for the study of recursive functions, which itself became a leading technique in the development of computers with Alan Turing (1912–1954), and has formed lasting links with both logic and mathematics (Davis 1965). Finally, the distinction between theory and meta-theory required by Gödel's proof brought home to logicians the care with which they needed to observe this distinction. For mathematicians, however, his theorems were of marginal importance, since Gödel worked with a much stricter concept of proof than that with which they were (or still are) accustomed.

However, most recent philosophy of arithmetic has built on versions of Hilbert's or on a set-theoretic approach; some have been constructivistic or intuitionistic in character. Some other treatments are nominalistic, in trying to avoid giving numbers an abstract status (Field 1987). In addition, social interpretations of numbers and arithmetic have gained some favour (Livingstone 1986), in which numbers are experienced, arithmetical operations are performed, and proofs are actions; "$2 + 2 = 4$" is less than a traditional Platonic truth, but more than a matter of personal opinion. Numbers are ancient things (or symbols, or sets, or acts, or ...); their statusses have always been obscure, and are likely to remain so.

Comments

1) Some aspects of formalism, especially recursion and computability, link nicely to under- and post-graduate courses in computing. But richer contexts arise in more elementary contexts, especially in stressing the difference between numbers and numerals, the brute sense of formalism. Relationships such as "$7 > 3$" should be studied carefully, and also the nonsense of wondering whether "3", "₃" or "**3**" is the true number three; for then the more abstract character of numbers relative to numerals can be clearly indicated, whatever status one chooses to assign to them.
2) In addition to numbers and numerals, *digit strings* should be stressed. We have long been familiar with them as telephone numbers, but now they come also in

barcodes, PIN numbers, and so on. While a digit string can easily be associated with its corresponding number (numerologists often do this, such as the orthodox Christians taking 111 as the Trinity number), they are treated differently, with much less mathematical content; in particular, they have no associated arithmetic. We even say strings in a different way ("one one one" rather than "one hundred and eleven"), and also write them; for example, there are various ways of stating telephone numbers in different countries.

3) There are a few overlaps; one is the manner of reading decimal expansions, where $27 \cdot 27$ is understood as "twenty seven point two seven" since the latter part is read from left to right. There is a psychological point involved here also; because of the geographical origins of HA, we read integers from right to left, contrary to that of the words of our language. We do this too often to notice, for integers up to the early millions; but when faced with, say, "463,563,640,863,759", then the reverse process becomes conscious, even with the partitioning of the component digits into threes to aid the reading.

9. References and further reading

The list does not attempt to be exhaustive. Many general books on the history of numbers are indifferent or worse as scholarly sources.

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