Isometries come in circles

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In the study of geometry, one is constantly confronted with groups of transformations on various "spaces." Many of these groups consist simply of the symmetries of those spaces with respect to suitably chosen properties. An obvious example is furnished by the symmetries of the cube. Geometrically speaking, these are the one-one transformations which preserve distances on the cube. They are known as "isometries," and are 48 in number.

[Birkhoff & MacLane, Survey of Modern Algebra (1941), p. 127]

The goal of this note is a purely geometrical classification of isometries of the line, circle, plane, sphere and space. The main idea employed is that every isometry of R^n is essentially determined, for n = 2 and n = 3 at least, by its effect on *one* circle, where it is conveniently 'trapped'. Moreover, we stress the affinity between isometries in R^1, R^2 and R^3 and offer, with the exception of one lemma, what we view as a *'two-dimensional'* classification of isometries of R^3. Our method differs substantially from the standard geometrical approach (as given, for example, in chapters 3 & 7 of [3] or 8 & 16 of [5]) and is, we hope, more constructive and intuitive.

Before we begin, let us recall that every isometry of R^1 is determined by its effect on 2 non-identical points, every isometry of R^2 is determined by its effect on 3 non-collinear points, every isometry of R^3 is determined by its effect on 4 non-coplanar points, and so on. This is a simple consequence of two well known facts:

(I) isometries of R^{n+1} map n-spheres to n-spheres of equal radius

(II) every n+2 n-spheres in R^{n+1} whose centers are 'in general position' can have at most one point in common; for example, the intersection of every 3 circles in R^2 the centers of which are not collinear is either a singleton or the empty set.

Isometries of R^1

Consider two distinct points W, E on a straight line L, with W 'west' of E and E 'east' of W. Given an isometry h, W′ = h(W) may of course be any point of L, including W itself. But, once W′ is known, the isometry condition |W′E′| = |WE| implies that there exist precisely two possibilities for E′ = h(E): either E′ lies east of W′ (*orientation preserved*), or E′ lies west of W′ (*orientation reversed*).

In the first case, |EE′| = |W′E′| – |W′E| = |WE| – |W′E| = |WW′| implies that E moves by the vector WW′, and so does, in view of |W′P′| = |WP| and |E′P′| = |EP|, *every* point P of L (figure 1): the isometry is a *translation* by WW′, reduced to the identity map (*trivial isometry*) in the special case W′ = W.

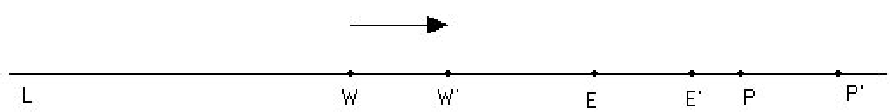


Fig. 1

In the second case, |WE′| = |W′E′| – |WW′| = |WE| – |WW′| = |W′E| implies that the midpoint M of WW′ is also the midpoint of EE′, and, thanks to |W′P′| = |WP| and |E′P′| = |EP| again, of PP′ for *every* point P of L as well (figure 2): the isometry is a *(point) reflection* about M.

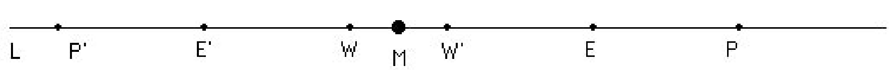


Fig. 2

<u>Proposition 1</u>: Every isometry of R^1 is either a translation or a reflection.

The following figure, precursor of two more interesting ones (11 and 17) to come, captures our observations:

Fig. 3

Isometries of S^1

Replacing line segments by circular arcs, we may employ the arguments of the previous section in order to show that every isometry of a circle is either a 'circular translation' or a 'circular point reflection'. Moreover, since equal circular arc lengths on a circle S^1 correspond to equal angles and equal straight line distances in R^2, it is both possible and convenient to view circular isometries as restrictions of planar isometries. Conversely, planar isometries may be viewed as 'unions' of circular isometries defined on a continuum of cocentric circles covering the plane.

Indeed a circular point reflection about M (or, equivalently, about M's *antidiametrical* point N) is clearly the restriction on that circle of a unique reflection about a line passing through M and the circle's center K (figure 4).

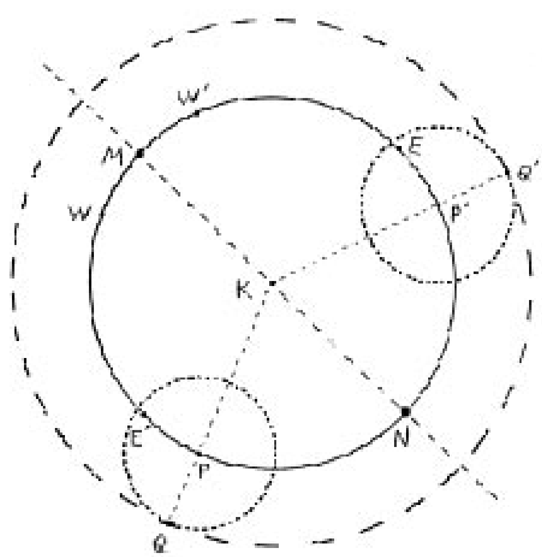


Fig. 4

Also, a circular translation by an oriented arc of length s on a circle of radius r may be viewed as the restriction on that circle of a unique *rotation* about K by a likewise oriented angle $\phi = s/r$ (figure 5), which is typically assumed to lie in $[0, \pi]$.

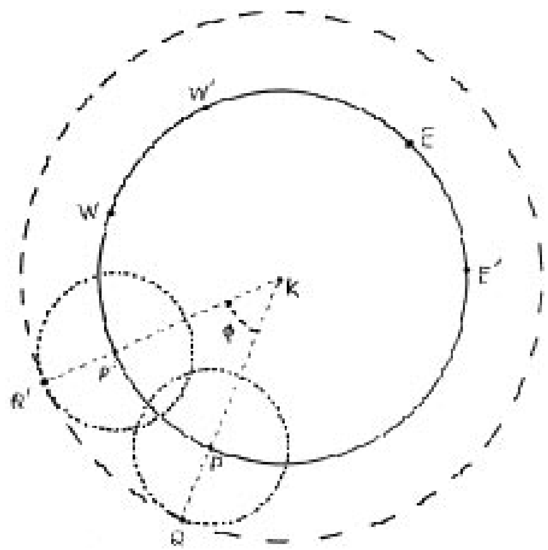


Fig. 5

<u>Proposition 2</u>: Every isometry of S^1 is either a reflection or a rotation.

Figures 4 and 5 illustrate how isometries of S^1 are extended to isometries of R^2 and how the image of an arbitrary point P is found ($P \to Q \to Q' \to P'$) using circles cocentric to the initial one and the fact that isometries preserve collinearity (of K, P, Q).

Isometries of R^2

Knowing that every isometry of R^2 is determined by its effect on three non-collinear points, we choose these three points to be two distinct, *non-antidiametrical* points W, E on a circle C plus the circle's center, K. Since isometries preserve circles, C is mapped to another circle C′ of equal radius, with W, E, K mapped to W′, E′, K′. Of course it is possible to have C′ = C (if and only if K′ = K), in which case there is nothing new to investigate: the isometry leaves C, and K as well, invariant, therefore (Proposition 2) it is either a rotation about K or a reflection about a line that passes through K.

Even in case K′ ≠ K, the translation by the vector K′K maps C′ back to C (C″ = C), K′ back to K (K″ = K) and W′, E′ to points W″, E″ on C. That is, we may view the isometry in question as the combined effect of a rotation or reflection (leaving C invariant and mapping W and E to W″ and E″) followed by the translation KK′ (figure 6). This combination produces two more possibilities for our planar isometry: composition of rotation followed by translation, and composition of reflection followed by translation.

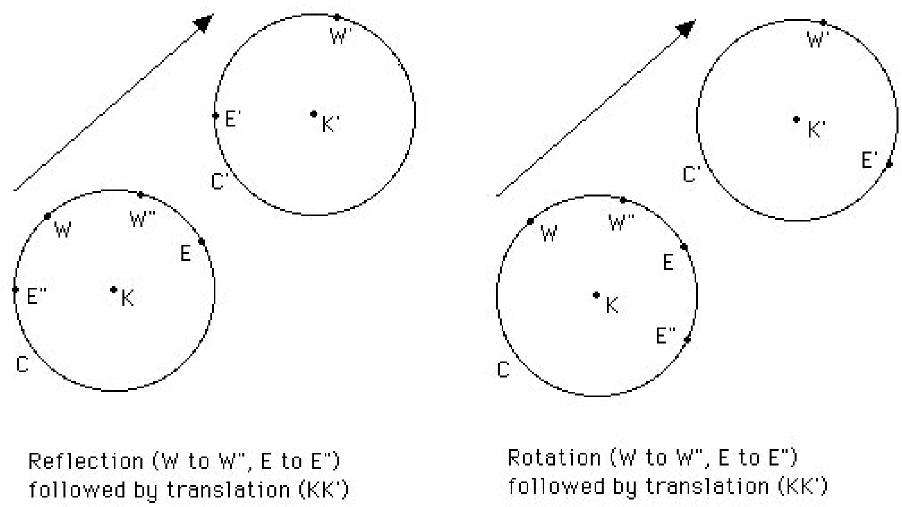


Fig. 6

The composition of a rotation followed by a translation is still a rotation, by the same angle but about a different center. To see this, observe first that every rotation by ϕ about K may be represented as the composition of any two reflections intersecting each other at K at an angle of $\phi/2$ (figure 7);

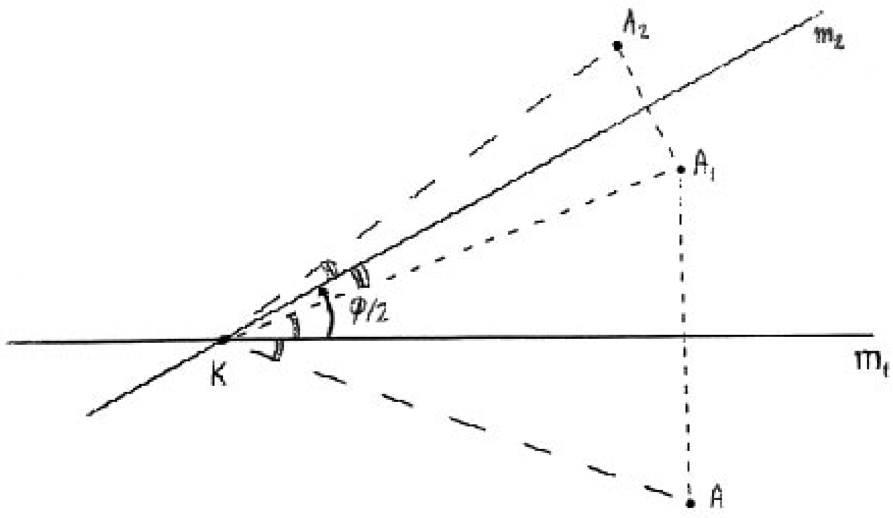


Fig. 7

as K moves toward infinity the two lines intersect each other further and further away, so it is not surprising that every translation of vector length d may be represented as the composition of any two parallel reflections at a distance d/2 from each other and perpendicular to the translation vector (figure 8).

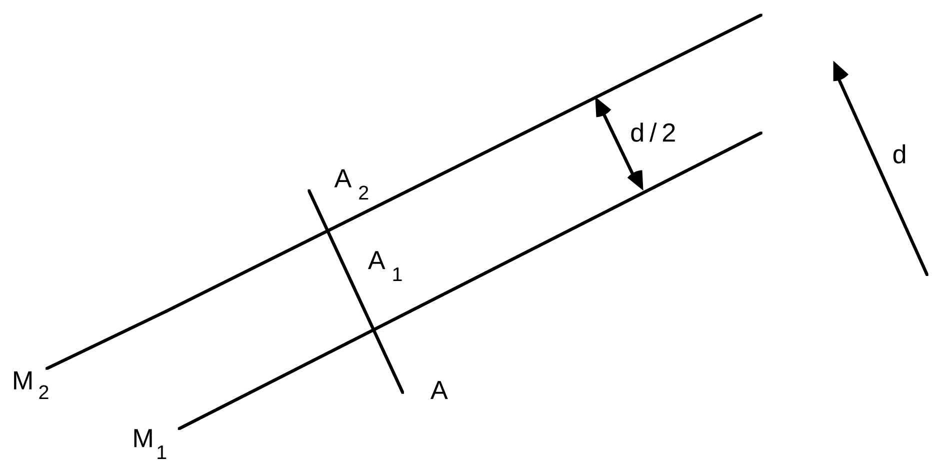


Fig. 8

Now to get the new rotation $r_2 = (K_2, \phi)$ out of the old rotation $r_1 = (K_1, \phi)$ followed by a translation t, all we need to do is choose r_1's ‘second’ reflection line to be the *same* as t’s ‘first’ reflection line: the square of the common reflection is the identity, so the net effect of the composition is the combination of the remaining two reflections, which must intersect each other at a point K_2 and at an angle $\phi/2$ (figure 9).

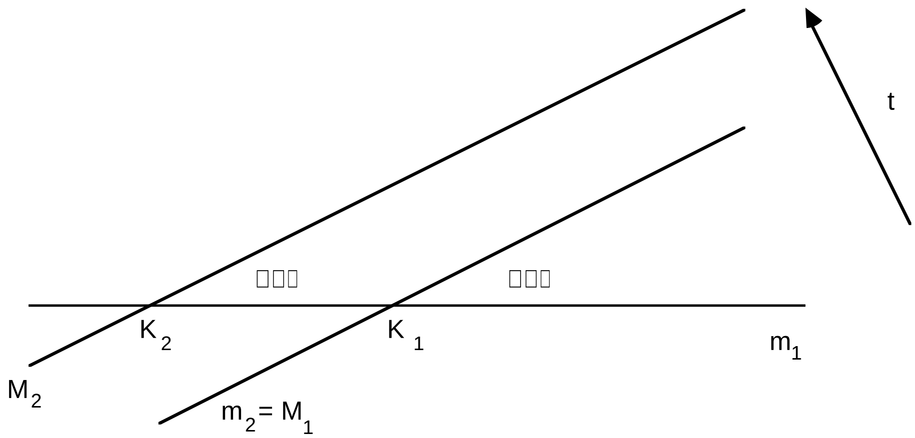


Fig. 9

We turn now to the composition of a reflection m followed by a translation t. Using the parallelogram rule, we analyse t into two translations, one perpendicular to m (t_1) and one parallel to m (t_2). Since t_1 may be expressed as the combination of m and a parallel to it reflection m′ (figure 10), the net effect of the composition is a reflection (m′) followed by a parallel to it translation (t_2): exactly as above, the square of m produced nothing but the identity.

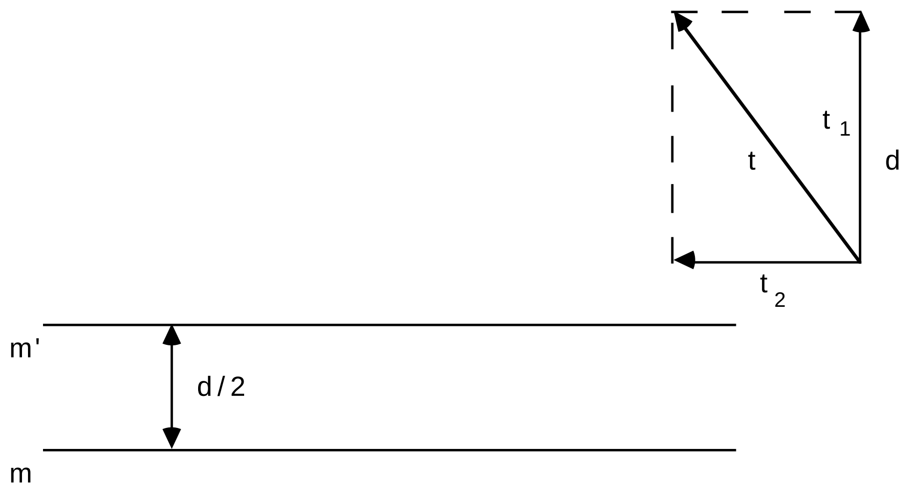


Fig. 10

Pairs of a reflection and a parallel to it translation (such as m′ and t_2) are important enough to be viewed as a single isometry known as *glide reflection*. For one thing, they do commute with each other, which is not generally the case for any two given isometries. Even more important, given two congruent sets on the plane, it is in general either a rotation or a glide reflection -- depending on the 'orientation' between them -- that maps one to the other: this is demonstrated in figure 11 below. (Of course a reflection may always be viewed as a glide reflection the translation vector of which has zero length; and a translation is sometimes viewed as a 'degenerate rotation' the center of which is the point at infinity.)

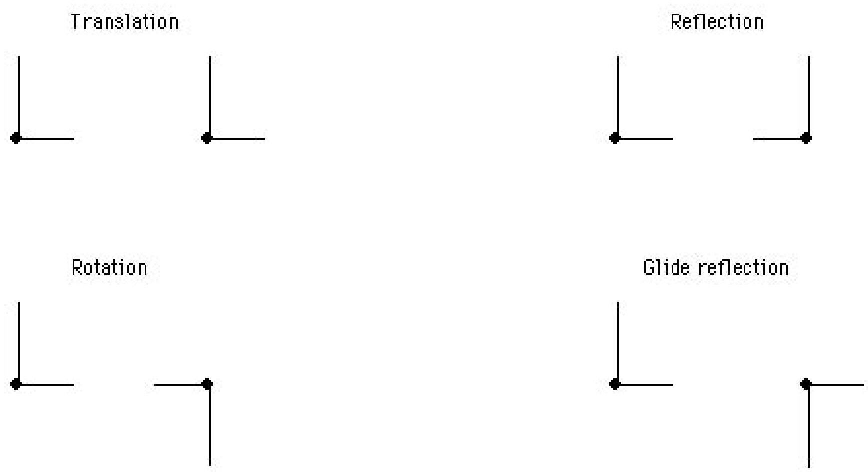


Fig. 11

Putting everything together, and keeping in mind the special case K′ ≠ K, W″ = W, E″ = E, which corresponds to a plain translation, we arrive at the following classification of planar isometries.

Proposition 3: Every isometry of R^2 is one of the following: translation, reflection, rotation, glide reflection.

Quite clearly in view of our analysis above, a rotation may occur either in case K′ = K or in case K′ ≠ K. Less obviously, the same is true of reflections: indeed the case K′ ≠ K occurs precisely when WW″ and EE″ are parallel to KK′, and corresponds to the composition of a reflection and a translation perpendicular to it (such as m and t_1 in figure 10, with $t_2 = 0$).

Notice that, unlike rotations and reflections, translations and glide reflections may not leave a finite set invariant. It follows that the isometry group of a polygon may only consist of rotations and reflections. In fact only 2 types of such groups are possible, the *cyclic* groups C_n and the *dihedral* groups D_n; see [5], section 8.2 for a proof of what is known among geometers as “Leonardo (da Vinci)’s Theorem”.

On the other hand, translations and glide reflections may well leave an *infinite* set invariant, as figure 12 illustrates:

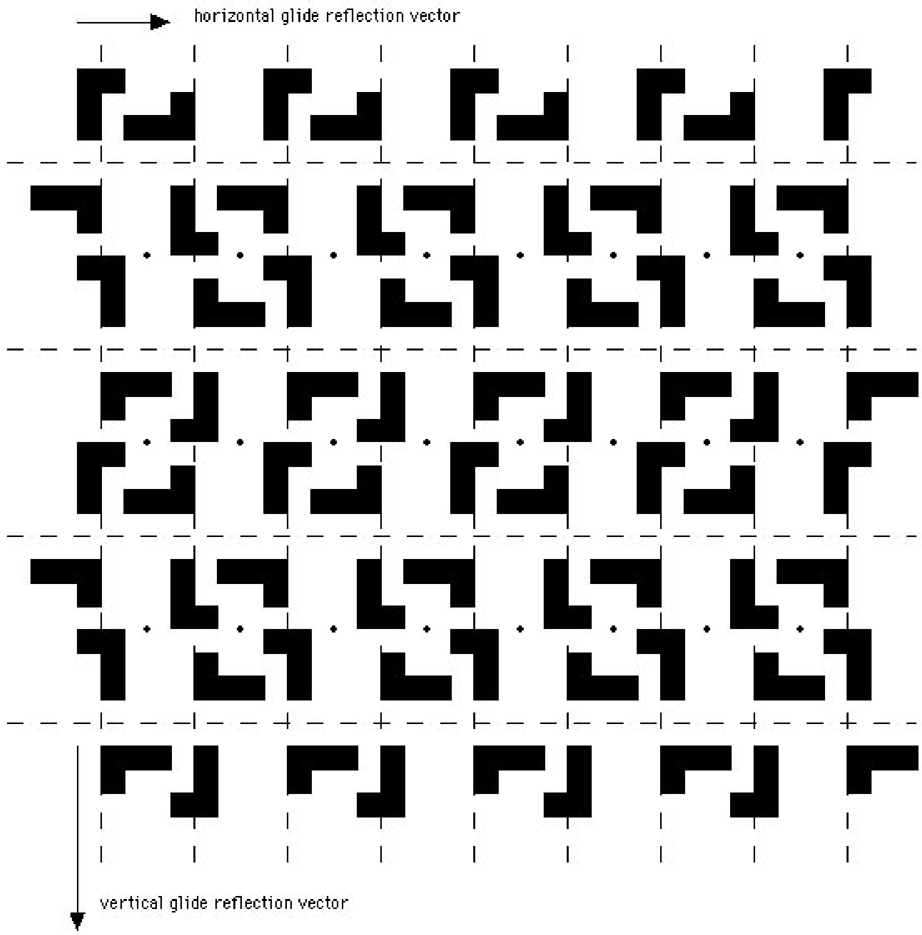


Fig. 12

The infinite set of figure 12 is a *wallpaper pattern* of pgg type, invariant under infinitely many glide reflections (represented by dotted lines) in two perpendicular directions. Depending on their group of isometries, there exist 17 distinct types of wallpaper patterns (i.e., sets invariant under translation in two distinct directions); a very accessible introduction to them is [6].

The pattern in figure 12 is also left invariant under the shown (centers of) 2-dimensional point reflections (i.e., 180^0 rotations): this is a consequence of the fact that the composition of two glide reflections intersecting each other at an angle ϕ is always a rotation by an angle 2ϕ about a point determined by the two translation vectors. This property of glide reflections is in full generality established by combining some of the tricks employed in this section. It is rather easier to show that the composition of two non-opposite rotations (K_1, ϕ_1), (K_2, ϕ_2) is again a rotation (K, ϕ):

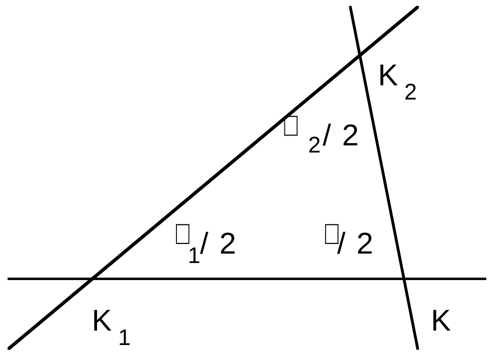


Fig. 13

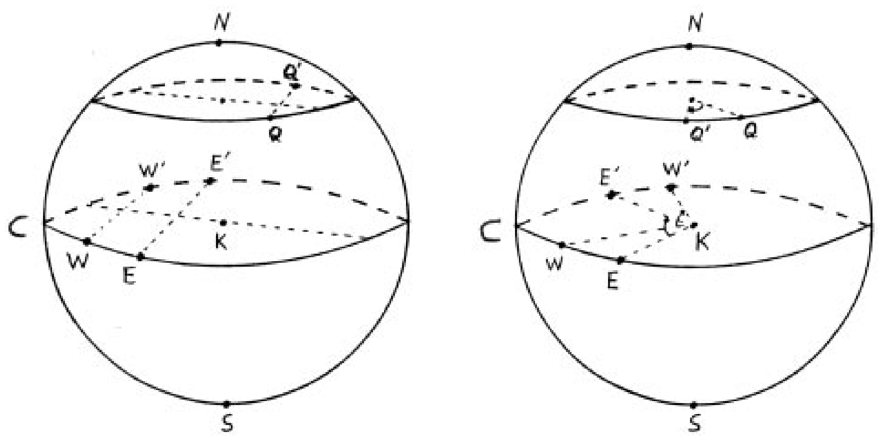
We have simply analysed each rotation into two reflections, and in such a way that the reflection K_1K_2 is shared by both rotations.

Isometries of S^2

In direct analogy to what happens in the case of S^1 and R^2, an isometry of S^2 is an isometry of R^3 that maps some sphere G to itself. Now of course we need four “generally positioned” points to determine such isometries, so we pick the center K of G, two non-antidiametrical points W, E on G’s “equator” (i.e., the great circle passing through W and E) and the “north pole” N. Clearly $K' = K$ holds for every isometry mapping G to itself. Now if in addition $N' = N$, then the equator has to be mapped to itself: indeed any “tilting” of the equator would lead to an obvious violation of at least one of the isometry conditions $|N'W'| = |NW|$ and $|N'E'| = |NE|$; in simpler terms, W' and E' may differ from W and E but they have to be on the equator. Conversely, the equator is mapped to itself ($C' = C$) only in case the north pole is either mapped to itself ($N' = N$) or mapped to the south pole ($N' = S$). So we need to investigate three cases altogether:

<u>Case 1:</u> $N' = N$, $C' = C$. The isometry’s restriction on C is a circular isometry, so that Proposition 2 applies, allowing precisely two

possibilities, a rotation and a reflection. But such a rotation and reflection must be “extendable” to isometries that in addition leave K and N invariant; in each case, the extending R^3 isometry has to be *unique*, due to its agreement with the original isometry on W, E, K and N. It is easy to see that the sought R^3 extensions are a *(line) rotation* about SN (extending (“level-by-level”) a 2-dimensional rotation about K) and a *(plane) reflection* about a plane perpendicular to C that bisects WE and contains SN (extending a 2-dimensional reflection about a line passing through K):



Reflection about a plane containing SN and perpendicular to WW', EE'

Rotation about SN by an oriented angle WKW' = EKE'

Fig. 14

<u>Case 2:</u> N′ = S, C′ = C. A reflection about C brings us to Case 1. Our R^3 isometry is therefore a composition of either reflections about C and a perpendicular to C plane containing SN or reflection about C and a rotation about SN. In the first case the outcome is easily seen to be a 180^0 rotation about the intersection line of the two planes, a line lying on C and passing through K. In the second case the outcome (composition of a reflection and a rotation perpendicular to each other) is a new 3-dimensional isometry called *rotoreflection* [2] or *rotation reflection* [4] or *rotary reflection* [5] or *rotatory reflection* [3]; we adopt the first term and discuss this isometry in some more detail further below.

<u>Case 3:</u> N′ ≠ N, N′ ≠ S, C′ ≠ C. This time W′ and E′ lie on C′, with the

arc W′E′ of C′ equal to the arc WE of C. Either a rotation about the intersection line of C, C′ or a reflection about the plane that bisects the angle of C, C′ maps C to C′, N to N′ and W, E to two points W″, E″ on C′. (Indeed those two isometries work, and there can be no others: any isometry mapping C to C′ (C″ = C′) must either satisfy N″ = N′ and S″ = S′ or N″ = S′ and S″ = N′; we pick the one that satisfies N″ = N′, be it the rotation or the reflection.) Once everything is on C′, we appeal to Case 1 to find either a “C′-reflection” (about a plane containing S′N′) or a “C′-rotation” (about S′N′) that maps W″ to W′ and E″ to E′ (and leaves N′ and S′ invariant). Putting everything together, we see (figure 15) that our isometry has to be one of the following:

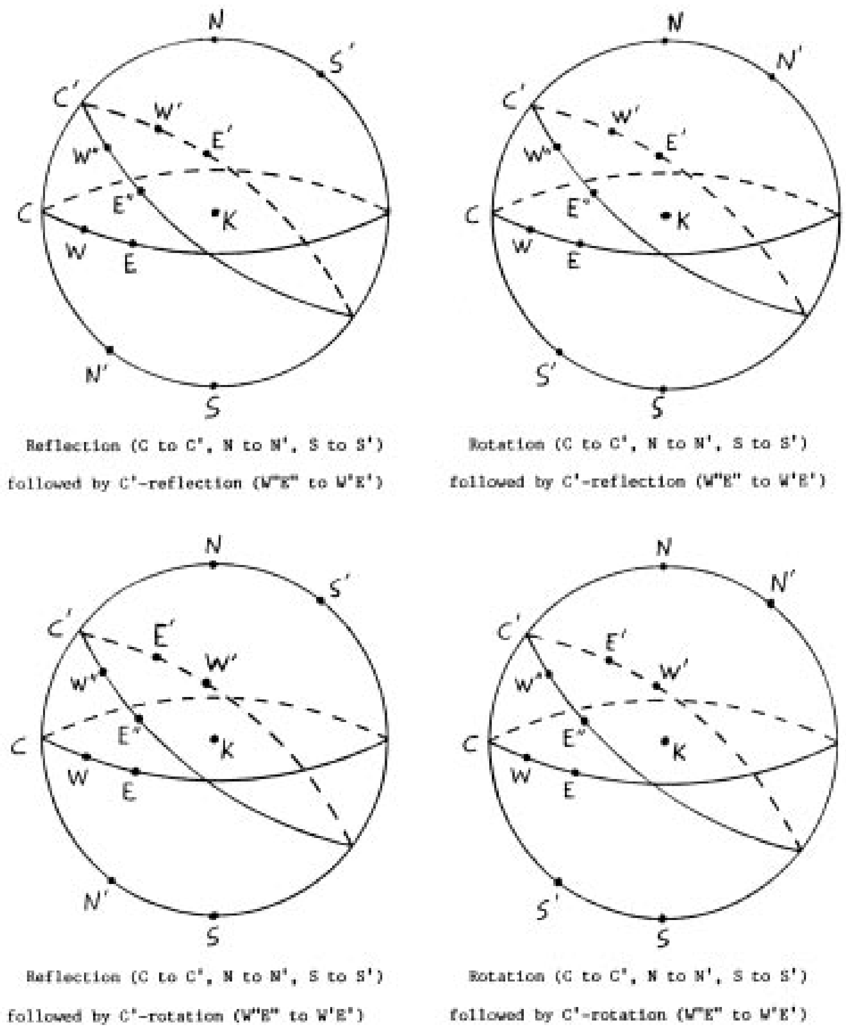


Fig. 15

-- composition of two reflections passing through K (which is a rotation about their intersection line by twice the angle between their planes, as figure 7's 3-dimensional version demonstrates)

-- composition of two rotations passing through K (which is again a rotation, as in figure 13's 3-dimensional version (employing the plane containing the two lines as "common reflection")

-- composition of a reflection P passing through K followed by a rotation L passing through K or vice versa: as our "3-dimensional lemma" at the end of this note shows, such a pair (P, L) may always be replaced by a pair (P_1, L_1) with P_1, L_1 perpendicular to each other and $P_1*L_1 = P*L$; in other words, the isometry in question in this last case is a rotoreflection. (Of course the rotoreflection may be just a reflection or rotation (satisfying N″ = N′) in the special case W″ = W′, E″ = E′ (no second isometry (leaving C′ invariant) needed).)

<u>Proposition 4:</u> Every isometry of S^2 is one of the following: reflection, rotation, rotoreflection.

In the case of rotoreflection, it is indeed significant that the reflection P_1 and the rotation L_1 are perpendicular to each other: that makes them commuting isometries, which is generally not the case between a rotation L and a reflection P. Of course something similar happens in R^2 with glide reflections: translation and reflection commute only in case they are parallel to each other. And, in the same way a commuting combination of reflection and translation in R^2 is "dynamic" enough to be viewed as an isometry on its own (glide reflection), the commuting combination of reflection and rotation in R^3 deserves a separate consideration and name (rotoreflection).

The similarity between glide reflection and rotoreflection pointed out above is of a rather deeper nature. Indeed, keeping the lattitude γ fixed, we may view the effect of a rotoreflection on a circle C_γ parallel to the equator C_0 as the composition of a "circular translation" (by an arc of length $(r\cos\gamma)\phi$) and a "circular reflection" about C_0 (figure 16). A connection between rotoreflection and glide reflection along similar lines is nicely illustrated in page 307 of [4].

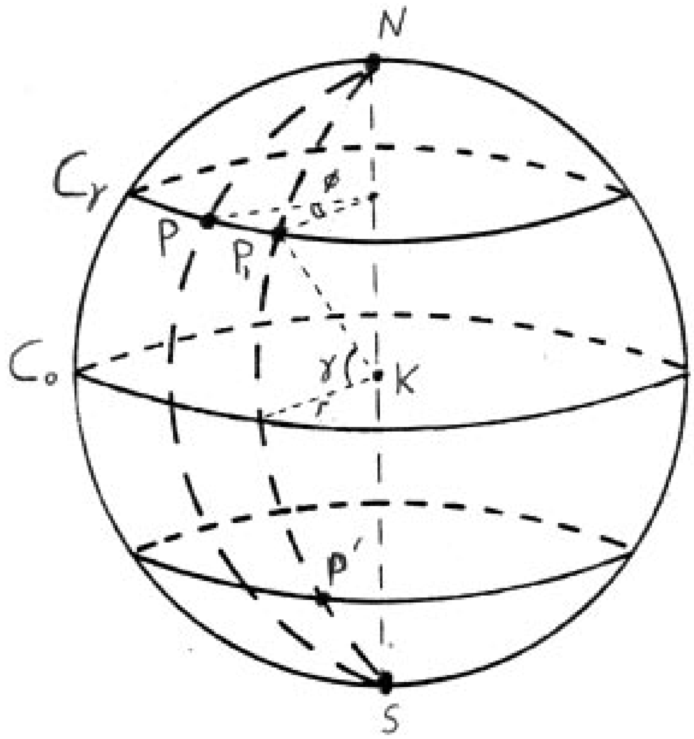


Fig. 16

Proposition 4 implies that the isometry group of a polyhedron may only consist of rotations, reflections and rotoreflections. In fact only 14 types of such groups are possible; see [5], section 17.2, for a proof of “Hessel’s Theorem” (which is effectively summarized in figure 8.28 of [4]). A closer look at the isometry group of the cube is provided in **http://www.oswego.edu/~baloglou/103/cube.html**, where the cube’s 48 isometries are classified into 10 types and a 10×10 “grouped group table” is provided.

Isometries of R^3

The derivation of the isometries of R^3 from the isometries of S^2 is entirely analogous to the derivation of the isometries of R^2 from the isometries of S^1, therefore we omit many details. Basically we end up composing an isometry inside a sphere G' with a translation (that maps G′ “back” to a sphere G). So we need to figure out the following compositions: reflection followed by translation, rotation followed by translation, and rotoreflection followed by translation.

For a reflection P followed by a translation T, all we need to do is to adapt figure 10 (and related comments) to R^3. In particular, we analyse the translation into one perpendicular to P (T_1) and one parallel to P (T_2); the final outcome is a reflection $P' = P*T_1$ parallel to the original one, followed by a parallel to it translation (T_2): predictably, this *commuting* combination of reflection and translation is still called a (3-dimensional) glide reflection.

Similarly, for a rotation L followed by a translation T, we combine figures 7, 8 and 9, adapted to R^3; but this time we analyse T into two components, one parallel to L (T_1) and one perpendicular to L (T_2). We end up with a new rotation $L' = T_2*L$ parallel to L (and of the same angle), followed by a parallel to it translation (T_1): this new isometry could or should be called *glide rotation* -- notice that L' and T_1 commute -- but is instead known in the existing literature as *screw (rotation)* or *twist* [3].

Finally, for a rotoreflection $P*L = L*P$ followed by a translation T, we again write T as T_1*T_2 (with T_1 and T_2 perpendicular to P and L, respectively) and we notice, employing ideas from the two cases above, that $(T_1*T_2)*(L*P) = T_1*(T_2*L)*P = T_1*L'*P = T_1*P*L' = (T_1*P)*L' = P'*L'$: the outcome is just another rotoreflection, the translation has been “absorbed”, no new isometry has been produced.

<u>Proposition 5:</u> Every isometry of R^3 is one of the following: translation, reflection, rotation, glide reflection, rotoreflection, glide rotation.

Our findings are visually summarized in figure 17, a natural companion to figures 3 and 11:

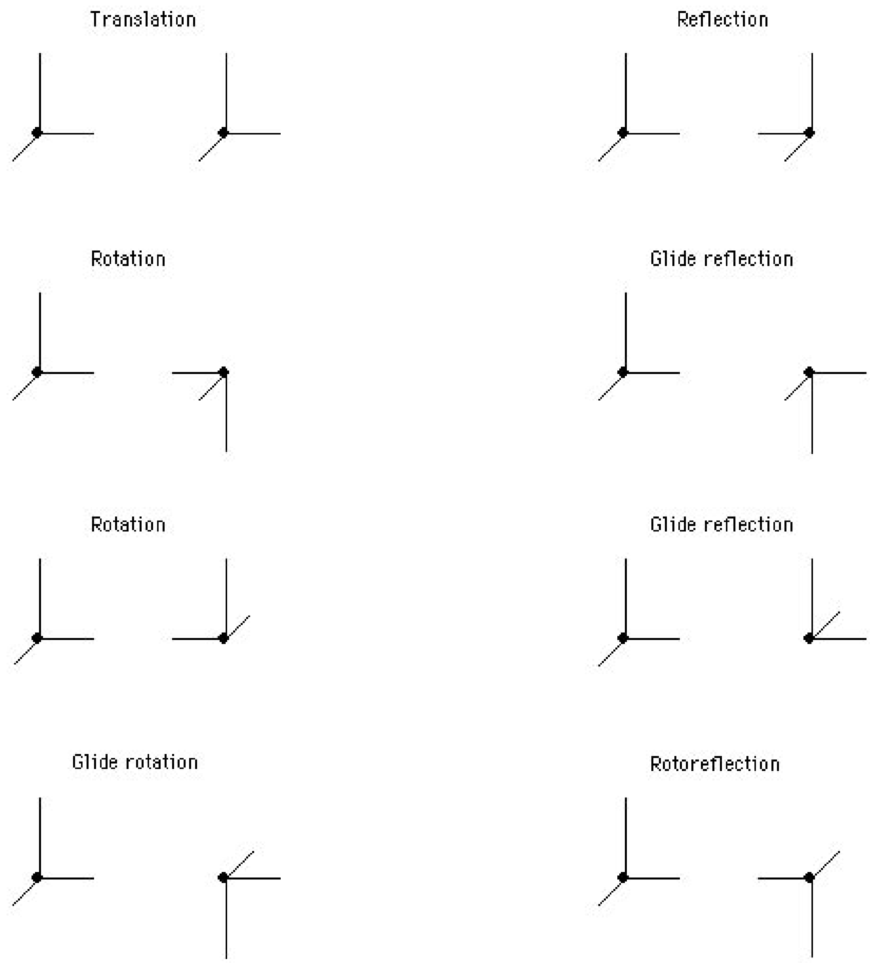


Fig. 17

The rotoreflection shown above, equal in fact to a 3-dimensional "point reflection", is special enough to be known under a separate name, *inversion* (about the midpoint of the two "origins"); it may be viewed (in *infinitely* many ways) as the composition of a reflection about *any* plane passing through that "midpoint") and a perpendicular to it 180^0 rotation. Crystallographers often use *rotoinversion* (composition of rotation and inversion) instead of rotoreflection: it is easy to see that a rotoinversion by an angle ϕ is equivalent to a rotoreflection by an angle $\phi-\pi$ (and about the same reflection plane).

Just as in figures 3 and 11, every pair on the left consists of two *'homostrophic'* sets (superimposable via mere sliding), while every pair on the right consists of two *'heterostrophic'* sets: this is not a coincidence, as reflection 'reverses orientation', and so do glide reflection and rotoreflection (products of three reflections); but products of two or four reflections (such as rotation/translation and glide rotation, respectively) are bound to 'preserve orientation'.

One should be careful when it comes to orientation. For example, a *planar* glide reflection may be viewed as the restriction of a 180^0 glide rotation: the former reverses orientation (in its own plane), the latter does not. Conversely, while a 180^0 rotoreflection reverses orientation in R^3, its restriction on any plane containing its axis is a 180^0 rotation (half turn) that preserves orientation (on its plane).

Just as in the case of R^2, no finite subset of R^3 may be invariant under translation, glide reflection or glide rotation. But infinite sets could be invariant under such isometries, and, just as there exist precisely 17 types of wallpaper patterns in R^2, there exist exactly 230 types of *crystallographic groups* in R^3; for example, just as figure 17 'extends' figure 11, there exist several 3-dimensional 'extensions' of the pattern in figure 12, based on copies of those 'triple L's. A good introduction to crystallographic groups is [2].

The compositions of 3-dimensional isometries are left to the reader to investigate: no new techniques beyond those presented here are needed. As an example, we mention that the composition of two glide reflections or skew rotations or glide rotations is a glide rotation. (In the case of skew rotations L_1 and L_2, write L_2*L_1 as $(P_2*S_2)*(S_1*P_1)$, where S_1, S_2 are the two *parallel* planes that contain L_1 and L_2.)

Rotoreflection = rotation * reflection

We finally discuss the promised 3-dimensional result that lies behind our "2-dimensional" classification of the isometries of R^3:

<u>Lemma</u>: The composition P*L of rotation about line L followed by reflection about plane P is always equal to some composition P_1*L_1 of rotation about line L_1 followed by reflection about plane P_1, with

L_1 perpendicular to P_1. Moreover, if γ is the angle between P and L, and ϕ is L's rotation angle, then

-- the angle between P and P_1 is $\tan^{-1}[(\tan(\phi/2))\cos\gamma]$

-- L_1's rotation angle is $\cos^{-1}\left[\dfrac{1+(\tan^2(\phi/2))\cos 2\gamma}{1+\tan^2(\phi/2)}\right]$.

Proof: Let L′ be L's projection on P, so that the angle between L and L′ equals γ, and let P′ be the unique plane that is perpendicular to both P and L′ and passes through their intersection O (figure 18). Let L_1, L_2 be the unique pair of lines on P′ satisfying $L(L_1) = L_2$ and, by necessity, being symmetric of each other about the plane defined by L and L′ (figure 18). Observe that the unique plane P_1 that passes through O and is perpendicular to L_1 is the only one that *might* work. Indeed $P*L(L_1) = P(L_2) = L_1$, and the only possibility for $P*L(L_1) = L_1$ with P and L perpendicular to each other is $P = P_1$ and $L = L_1$: the only line left invariant under a rotoreflection $P*L$ is its own "axis".

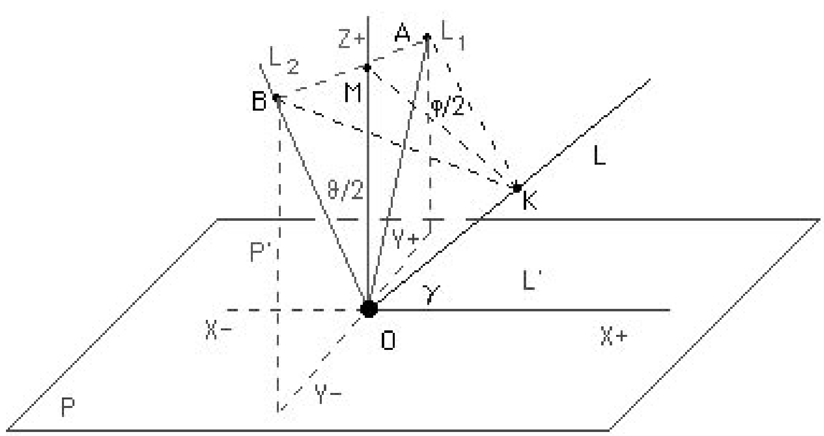


Fig. 18

To make P_1*L_1 work, it suffices to pick a rotation angle for L_1 so that P_1*L_1 will be equal to P*L on just one point outside L_1. Indeed P_1*L_1 and P*L agree on O and every single point on L_1; and every two rotoreflections that are equal to each other on three non-collinear points must equal each other everywhere on R^3. To see this, observe first that if the isometries I_1 and I_2 of R^3 are equal on three non-collinear points A, B, C then either $I_1 = I_2$ or $I_1 = m*I_2$, where m is the reflection about the plane defined by A, B, C. Indeed for every point D outside the plane ABC, both $I_1(D)$ and $I_2(D)$ must lie on the three spheres (A, |AD|), (B, |BD|) and (C, |CD|), which have precisely two points in common, symmetric of each other about ABC; so either I_2 or $m*I_2$ agrees with I_1 on the four non-coplanar points A, B, C, D (hence everywhere else). But the composition of a rotoreflection and a reflection (rotation * reflection * reflection) is easily seen to be either a twist or a rotation -- the latter in our case, as P_1 does pass through the intersection of P and L -- which, unlike rotoreflections, do "preserve orientation" (see figure 17 and related comments): this rules out $I_1 = m*I_2$ in our case ($I_1 = P_1*L_1$, $I_2 = P*L$).

For convenience, we determine L_1's angle ϕ_1 so that P_1*L_1 agrees with P*L on the point K on L. First, we set our coordinate axes: the x-axis will be L′, the y-axis will be the intersection of P′ with P, and the z-axis will be the projection of L on P′, which is of course perpendicular to P (figure 18). Next, notice that the angle between P and P_1 is equal to that between the z-axis (OM) and L_1 (OA), which is of course half the angle θ between L_1 and L_2. Finally, L(A) = B (hence $\angle AKB = \phi$) and $\angle KOM = \pi/2-\gamma$, together with simple trigonometry in the triangles MAK (right angle at M), MAO (right angle at M) and KOM (right angle at K) establish the relations $\tan(\phi/2) = |AM|/|KM|$, $\tan(\theta/2) = |AM|/|OM|$ and $\cos\gamma = |KM|/|OM|$ (figure 18); elimination then yields $\tan(\theta/2) = (\tan(\phi/2))\cos\gamma$.

Set now |OK| = d, so that $K = (d\cos\gamma, 0, d\sin\gamma)$. Clearly, $P*L(K) = P(K) = K' = (d\cos\gamma, 0, -d\sin\gamma)$. Since P_1 and L_1 are perpendicular they commute, hence $P_1*L_1(K) = L_1*P_1(K) = L_1(K'_1)$, where K'_1 is the image of K under reflection about the line $z = -\tan(\theta/2)y$ on the plane $x = d\cos\gamma$; as figure 19 shows, reflection about that line maps (0, h) to $(-h\sin\theta, -h\cos\theta)$, therefore $K'_1 = (d\cos\gamma, -d\sin\gamma\sin\theta, -d\sin\gamma\cos\theta)$ and $|K'K'_1| = 2d\sin\gamma\sin(\theta/2)$.

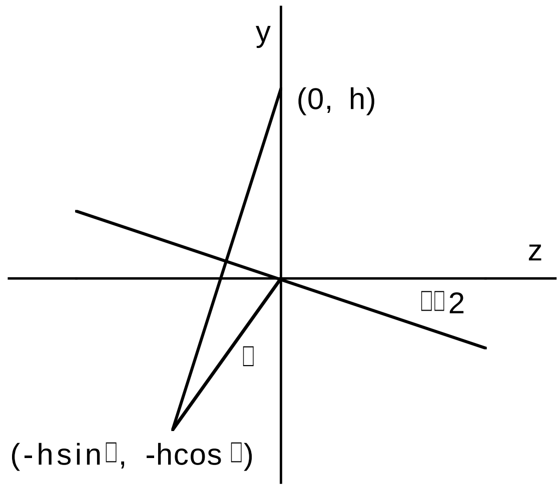


Fig. 19

Now we need to pick ϕ_1 so that $L_1(K'_1) = K'$. Observe that L_1 may be written as $\{x = 0,\ z = (\cot(\theta/2))y\}$. Ordinary Calculus shows that the minimum distance of (a, b, c) from $\{x = 0,\ z = my\}$ occurs at $y = \frac{mc+b}{m^2+1}$ and equals $\sqrt{a^2+\frac{(mb-c)^2}{m^2+1}}$. It is easy then to check that the minimum distances of *both* K' and K'_1 from L_1 occur at $y = \frac{-d\sin\gamma\sin\theta}{2}$, for a point N on L_1, and are equal: $|K'N| = |K'_1N| = d\sqrt{1-\sin^2\gamma\cos^2(\theta/2)}$. Applied to $K'K'_1N$, the law of cosines yields, in view of $\tan(\theta/2) = (\tan(\phi/2))\cos\gamma$, $\cos\phi_1 = \left[\frac{1+(\tan^2(\phi/2))\cos 2\gamma}{1+\tan^2(\phi/2)}\right]$.

A non-analytical “derivation” of N, easily extendable (by way of $\tan(\theta/2) = (\tan(\phi/2))\cos\gamma$) to a Calculus-free computation of ϕ_1, is shown below:

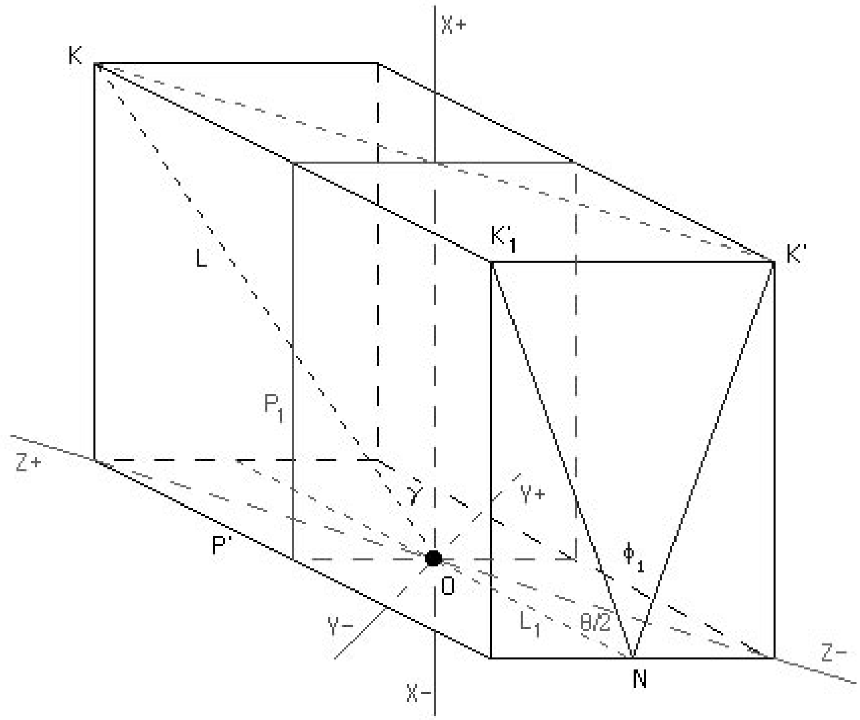


Fig. 20

Remarks:

(1) It follows from the main rotoreflection angle formula that $0 < \phi \leq \pi$ yields $0 < \phi_1 \leq 2\gamma$, with $\phi_1 < \phi$.

(2) If ϕ is replaced by $-\phi$, or P*L by L*P, then L_1 is replaced by L_2.

(3) The determination of L_1 and P_1 is illustrated through six examples in **http://www.oswego.edu/~baloglou/103/rotref.html**; those examples involve various pairs of reflection and rotation in a cube.

This note is an outgrowth of work in progress devoted to wallpaper patterns [1]. I would like to thank my colleagues Patrick Halpin and Kathleen Lewis for some helpful remarks, and several other colleagues and students of “Symmetries” at SUNY Oswego for the support and inspiration they have provided.

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