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SPHERES IN A WEYL SPACE

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ABSTRACT. In this paper, we have defined spheres in a Weyl space and obtained their characterization.

1. INTRODUCTION

A differentiable manifold of dimension n having a conformal metric tensor g and a symmetric connection ∇ satisfying the compatibility condition

$$\nabla g = 2(T \otimes g) \tag{1.1}$$

where T is a $1-$form(complementary covector field) is called a Weyl space which we denote it by $W_n(g, T)$. Under the conformal change

$$\bar{g} = \lambda^2 g \tag{1.2}$$

of the metric tensor g, T is transformed by the law

$$\bar{T} = T + d \ln \lambda. \tag{1.3}$$

An object A defined on $W_n(g, T)$ is called a satellite of g of weight $\{p\}$ if it admits a transformation of the form

$$\bar{A} = \lambda^p A \tag{1.4}$$

under the conformal change of g([1],[2],[3]). The prolonged derivative and the prolonged covariant derivative in the direction of the vector X of the satellite A of weight $\{p\}$ are, respectively defined by

$$\dot{\partial}_X A = \partial_X A - pT(X)A \tag{1.5}$$

and

$$\dot{\nabla}_X A = \nabla_X A - pT(X)A \tag{1.6}$$

where $\partial_X A$ is the derivative of A in the direction of X. By (1.2) and (1.6) it follows that for every X, $\dot{\nabla}_X g = 0$.

We note that prolonged differentiation and prolonged covariant differentiation preserve the weights of the satellites.

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Let $x \in U \subset W_n(g,T)$, $X \in T_x(U)$, $A \in \chi(U)$, and let $X = \sum_{k=1}^n X^k(\frac{\partial}{\partial x^k})_x$, $A = \sum_{i=1}^n A^i \frac{\partial}{\partial x^i}$, $T = \sum_{k=1}^n T_k dx^k$. Then (1.6) gives

$$X^k \dot{\nabla}_k A^i = X^k \nabla_k A^i - pT_k A^i \ , \ \nabla_k = \nabla_{\frac{\partial}{\partial x^k}} . \tag{1.7}$$

Spheres in Riemannian spaces are extensively studied by K.Yano and K.Nomizu [4].

The definition of a sphere given in a Riemannian space is not applicable in a Weyl space since the length of a vector field is not a gauge invariant. In the following we use the prolonged covariant differentiation in the definition of a sphere in a Weyl space due to the fact that it preserves the metric tensor and the weights of the satellites.

As far as we know, spheres in Weyl spaces have not yet been studied.

Let $W_n(g,T)$ be a subspace of the Weyl space $\bar{W}_m(\bar{g},\bar{T})$ and let ∇ and $\bar{\nabla}$ be the corresponding connections. Let $p\varepsilon W_n(g,T)$ and let U, $\bar{U}$ be the special coordinate neighborhoods of p. Then, the Gauss equation and the Weingarten equation for $W_n(g,T)$ are respectively

$$\bar{\nabla}_{\bar{X}}\bar{Y}|_U = \bar{\nabla}_X Y = \nabla_X Y + \alpha(X,Y) \tag{1.8}$$

$$\bar{\nabla}_{\bar{X}}\bar{\xi}|_U = \bar{\nabla}_X \xi = -A_\xi X + \nabla^\perp_X \xi \tag{1.9}$$

where $X\varepsilon T_p(U)$, $Y\varepsilon\chi(U)$ and ξ is a vector field normal to $W_n(g,T)$ while $\bar{X}$, $\bar{Y}$ are extensions of X ve Y to $\bar{U}$[5].

We now find the expressions for Gauss and Weingarten equations in terms of prolonged covariant derivative. The prolonged covariant derivative of the vector field $Y\varepsilon\chi(U)$ of weight $\{-1\}$ in the direction of X is, according to (1.6)

$$\dot{\bar{\nabla}}_X Y = \bar{\nabla}_X Y + \bar{T}(X)Y. \tag{1.10}$$

By (1.8), (1.10) becomes $\dot{\bar{\nabla}}_X Y = \nabla_X Y + \alpha(X,Y) + \bar{T}(X)Y$ from which it follows that

$$\begin{aligned} tan\dot{\bar{\nabla}}_X Y &= \nabla_X Y + \bar{T}(X)Y = \dot{\nabla}_X Y \\ nor\dot{\bar{\nabla}}_X Y &= \alpha(X,Y) \end{aligned}$$

and consequently we have

$$\begin{aligned} \dot{\bar{\nabla}}_X Y &= tan\dot{\bar{\nabla}}_X Y + nor\dot{\bar{\nabla}}_X Y \\ \dot{\bar{\nabla}}_X Y &= \dot{\nabla}_X Y + \alpha(X,Y). \end{aligned} \tag{1.11}$$

Similarly, the normal vector field ξ of weight $\{-1\}$ has the prolonged covariant derivative

$$\dot{\bar{\nabla}}_X \xi = \bar{\nabla}_X \xi + \bar{T}(X)\xi \tag{1.12}$$

in the direction of X. By the Weingarten equation (1.9), (1.12) takes the form

$$\dot{\bar{\nabla}}_X \xi = -A_\xi X + \nabla^\perp_X \xi + \bar{T}(X)\xi.$$

Since

$$
\begin{aligned}
tan\dot{\bar{\nabla}}_X\xi &= -A_\xi X, \\
nor\dot{\bar{\nabla}}_X\xi &= \nabla^\perp_X\xi + \bar{T}(X)\xi,
\end{aligned}
\tag{1.13}
$$

(1.12) reduces to

$$
\begin{aligned}
\dot{\bar{\nabla}}_X\xi &= tan\dot{\bar{\nabla}}_X\xi + nor\dot{\bar{\nabla}}_X\xi \\
\dot{\bar{\nabla}}_X\xi &= -A_\xi X + \nabla^\perp_X\xi + \bar{T}(X)\xi,
\end{aligned}
$$

or

$$
\dot{\bar{\nabla}}_X\xi = -A_\xi X + \dot{\nabla}^\perp_X\xi. \tag{1.14}
$$

A normal vector field ξ on $W_n(g,T)$ is said to be parallel with respect to $\nabla^\perp_X$ if

$$
\nabla^\perp_X\xi = 0 \tag{1.15}
$$

for every tangent vector X.

The vector field

$$
\eta = \frac{1}{n}tr\alpha \tag{1.16}
$$

is called the mean curvature vector of $W_n(g,T)$.

Let C be a smooth curve belonging to the Weyl space $W_n(g,T)$ and let $\underset{1}{\xi}$ be the tangent vector to C at the point P normalized by the condition $g(\underset{1}{\xi},\underset{1}{\xi}) = 1$. A curve in $W_n(g,T)$ is called a circle if there exist a vector field $\underset{2}{\xi}$, normalized by the condition $g(\underset{2}{\xi},\underset{2}{\xi}) = 1$, along C and a positive prolonged covariant constant scalar function κ of weight $\{-1\}$ such that

$$
\dot{\nabla}_{\underset{1}{\xi}}\underset{1}{\xi} = \kappa\underset{2}{\xi} \tag{1.17}
$$

$$
\dot{\nabla}_{\underset{1}{\xi}}\underset{2}{\xi} = -\kappa\underset{1}{\xi}. \tag{1.18}
$$

We note that the equations (1.17) and (1.18) are invariant under a gauge transformation.

A circle C satisfies the third order differential equation

$$
\dot{\nabla}^2_{\underset{1}{\xi}}\underset{1}{\xi} + g(\dot{\nabla}_{\underset{1}{\xi}}\underset{1}{\xi}, \dot{\nabla}_{\underset{1}{\xi}}\underset{1}{\xi})\underset{1}{\xi} = 0\ ,\ \underset{1}{\xi}^i = \frac{dx^i}{ds} \tag{1.19}
$$

where s is the arclength of C measured from a fixed point on C and x^i are the coordinates of a current point belonging to C [6].

2. SPHERES IN A WEYL SPACE

Let $W_n(g,T)$ be a submanifold of a Weyl space $\bar{W}_m(\bar{g},\bar{T})$. In this section the concept of a sphere in Riemannian space will be generalized to a Weyl space.

Definition 1. *If $W_n(g,T)$ is an n-dimensional umbilical submanifold of the m-dimensional Weyl space $\bar{W}_m(\bar{g},\bar{T})$ with a non-zero curvature vector then it is called an extrinsic sphere or simply a sphere.*

Theorem 1. *Let $W_n(g,T)$ $(n \geq 2)$ be a submanifold of the Weyl space $\bar{W}_m(\bar{g},\bar{T})$. If every circle in $W_n(g,T)$ is a circle in $\bar{W}_m(\bar{g},\bar{T})$, then $W_n(g,T)$ is a sphere in $\bar{W}_m(\bar{g},\bar{T})$. Conversely, if $W_n(g,T)$ is a sphere in $\bar{W}_m(\bar{g},\bar{T})$, then every circle in $W_n(g,T)$ is a circle in $\bar{W}_m(\bar{g},\bar{T})$.*

Proof. Let x be an arbitrary point of $W_n(g,T)$ and ξ, η orthonormal vectors in the tangent space of $W_n(g,T)$ at x. Let C be a curve admitting a parametric representation $x = x(s)$. We now take the arc length $(|s| < \varepsilon)$ as a parameter for the curve C. Let the curve C be a circle in the submanifold $W_n(g,T)$ such that

$$x(0) = x_0 = x\ ,\ \xi(0) = \xi_0 = \xi\ ,\ \dot{\nabla}_{\underset{1}{\xi}}\underset{1}{\xi}\ |_0 = \frac{1}{r}\eta \tag{2.1}$$

where $\underset{1}{\xi}$ is a tangent vector of the circle C.

Since C is a circle, the vector $\underset{1}{\xi}$ satisfies the differential equation

$$\dot{\nabla}^2_{\underset{1}{\xi}}\underset{1}{\xi} + g(\dot{\nabla}_{\underset{1}{\xi}}\underset{1}{\xi}, \dot{\nabla}_{\underset{1}{\xi}}\underset{1}{\xi})\underset{1}{\xi} = 0. \tag{2.2}$$

Denoting by $\bar{\nabla}$ the connection of $\bar{W}_m(\bar{g},\bar{T})$, if the curve C in $W_n(g,T)$ is a circle in $\bar{W}_m(\bar{g},\bar{T})$, then we have

$$\dot{\bar{\nabla}}^2_{\underset{1}{\xi}}\underset{1}{\xi} + g(\dot{\bar{\nabla}}_{\underset{1}{\xi}}\underset{1}{\xi}, \dot{\bar{\nabla}}_{\underset{1}{\xi}}\underset{1}{\xi})\underset{1}{\xi} = 0. \tag{2.3}$$

Denoting by α the second fundamental form of $W_n(g,T)$ and using the Gauss equation (1.11) we get

$$\dot{\bar{\nabla}}_{\underset{1}{\xi}}\underset{1}{\xi} = \dot{\nabla}_{\underset{1}{\xi}}\underset{1}{\xi} + \alpha(\underset{1}{\xi},\underset{1}{\xi}). \tag{2.4}$$

Taking the prolonged covariant derivative of (2.4) in the direction of $\underset{1}{\xi}$ we obtain

$$\begin{aligned} \dot{\bar{\nabla}}^2_{\underset{1}{\xi}}\underset{1}{\xi} &= \dot{\bar{\nabla}}_{\underset{1}{\xi}}(\dot{\nabla}_{\underset{1}{\xi}}\underset{1}{\xi}) + \dot{\bar{\nabla}}_{\underset{1}{\xi}}\alpha(\underset{1}{\xi},\underset{1}{\xi}) \\ &= \dot{\nabla}^2_{\underset{1}{\xi}}\underset{1}{\xi} + \alpha(\underset{1}{\xi}, \dot{\nabla}_{\underset{1}{\xi}}\underset{1}{\xi}) - A_{\alpha(\underset{1}{\xi},\underset{1}{\xi})}\underset{1}{\xi} + \dot{\nabla}^{\perp}_{\underset{1}{\xi}}\alpha(\underset{1}{\xi},\underset{1}{\xi}) \end{aligned} \tag{2.5}$$

where A_ξ is the shape operator for a normal vector ξ and $\dot{\nabla}^{\perp}$ denotes the prolonged covariant differentiation along C relative to the normal connection. Substituting (2.4) and (2.5) into (2.3) and taking account of (2.2) we get

$$\alpha(\underset{1}{\xi}, \dot{\nabla}_{\underset{1}{\xi}}\underset{1}{\xi}) - A_{\alpha(\underset{1}{\xi},\underset{1}{\xi})}\underset{1}{\xi} + \dot{\nabla}^{\perp}_{\underset{1}{\xi}}\alpha(\underset{1}{\xi},\underset{1}{\xi}) + g(\alpha(\underset{1}{\xi},\underset{1}{\xi}), \alpha(\underset{1}{\xi},\underset{1}{\xi}))\underset{1}{\xi} = 0. \tag{2.6}$$

Separating (2.6) into tangential and normal components , we get respectively

$$A_{\alpha(\underset{1}{\xi},\underset{1}{\xi})}\underset{1}{\xi} = g(\alpha(\underset{1}{\xi},\underset{1}{\xi}),\alpha(\underset{1}{\xi},\underset{1}{\xi}))\underset{1}{\xi} \tag{2.7}$$

and

$$\alpha(\underset{1}{\xi},\dot{\nabla}_{\underset{1}{\xi}}\underset{1}{\xi}) + \dot{\nabla}^{\perp}_{\underset{1}{\xi}}\alpha(\underset{1}{\xi},\underset{1}{\xi}) = 0. \tag{2.8}$$

Using

$$(\dot{\nabla}^{*}_{\underset{1}{\xi}}\alpha)(\underset{1}{\xi},\underset{1}{\xi}) = \dot{\nabla}^{\perp}_{\underset{1}{\xi}}\alpha(\underset{1}{\xi},\underset{1}{\xi}) - 2\alpha(\dot{\nabla}_{\underset{1}{\xi}}\underset{1}{\xi},\underset{1}{\xi}) \tag{2.9}$$

we may rewrite (2.8) in the form

$$3\alpha(\dot{\nabla}_{\underset{1}{\xi}}\underset{1}{\xi},\underset{1}{\xi}) = -(\dot{\nabla}^{*}_{\underset{1}{\xi}}\alpha)(\underset{1}{\xi},\underset{1}{\xi}) \ , \tag{2.10}$$

where $\dot{\nabla}^{*}_{\underset{1}{\xi}}\alpha$ is the Weyl version of the natural covariant derivative of the second fundamental form α in the direction of $\underset{1}{\xi}$ in the Riemannian case [7].

Noting that $\dot{\nabla}_{\underset{1}{\xi}}\underset{1}{\xi}\mid_0 = \frac{1}{r}\eta$ and using (2.2) we obtain

$$\alpha(\eta,\xi) = -\frac{r}{3}(\dot{\nabla}^{*}_{\underset{1}{\xi}}\alpha)(\xi,\xi). \tag{2.11}$$

This equation shows that $\alpha(\eta,\xi)$ is independent of η provided that η is orthogonal to ξ. In particular changing η into $-\eta$ we find $\alpha(\eta,\xi) = 0$.

To complete the proof of the theorem we now establish the

Lemma Let x be an arbitrary point of $W_n(g,T)$, and let ξ, η be orthonormal vectors in the tangent space of $W_n(g,T)$ at x so that $\alpha(\xi,\eta) = 0$. Then the following conclusions hold:

(1) $\alpha(\xi,\xi) = \alpha(\eta,\eta)$ for any orthonormal ξ and η in $T_x(W_n)$.

(2) The mean curvature vector η_x is equal to $\alpha(\underset{1}{\xi},\underset{1}{\xi})$, where $\underset{1}{\xi}$ is an arbitrary vector in $T_x(W_n)$ normalized by the condition $g(\underset{1}{\xi},\underset{1}{\xi}) = 1$.

(3) $W_n(g,T)$ is umbilical at x , i.e.

$$\alpha(\xi,\eta) = g(\xi,\eta)\eta_x \ , \qquad \text{for all } \ \xi,\eta \in T_x(W_n).$$

Proof.

(1) Since ξ and η are orthonormal, so are $\frac{1}{\sqrt{2}}(\xi+\eta)$ and $\frac{1}{\sqrt{2}}(\xi-\eta)$. Thus

$$\alpha(\frac{1}{\sqrt{2}}(\xi+\eta),\frac{1}{\sqrt{2}}(\xi-\eta)) = 0$$

which implies that

$$\alpha(\xi,\xi) = \alpha(\eta,\eta).$$

(2) Let $\{\underset{1}{\xi},\underset{2}{\xi},...,\underset{n}{\xi}\}$ be an orthonormal basis in $T_x(W_n)$.By (1)

$$\alpha(\underset{1}{\xi},\underset{1}{\xi}) = \alpha(\underset{2}{\xi},\underset{2}{\xi}) = ... = \alpha(\underset{n}{\xi},\underset{n}{\xi}).$$

Therefore we find

$$\eta_x = \frac{1}{n}\sum_{i=1}^{n}\alpha(\underset{i}{\xi},\underset{i}{\xi}) = \alpha(\underset{1}{\xi},\underset{1}{\xi}).$$

(3) Since $\{\underset{1}{\xi},\underset{2}{\xi},...,\underset{n}{\xi}\}$ is an orthonormal basis we can write

$$\xi = \sum_{i=1}^{n} a_i\underset{i}{\xi} \; and \; \eta = \sum_{i=1}^{n} b_i\underset{i}{\xi}.$$

So that

$$\begin{aligned}\alpha(\xi,\eta) &= \sum_{i,j=1}^{n} a_i b_j \alpha(\underset{i}{\xi},\underset{j}{\xi}) = (\sum_{i=1}^{n} a_i b_i)\alpha(\underset{1}{\xi},\underset{1}{\xi}) \\ &= g(\xi,\eta)\eta_x.\end{aligned}$$

Now we go back to the theorem. Since $g(\underset{1}{\xi},\dot{\nabla}_{\underset{1}{\xi}}\underset{1}{\xi}) = 0$ we have $\alpha(\underset{1}{\xi},\dot{\nabla}_{\underset{1}{\xi}}\underset{1}{\xi}) = 0$. Thus (2.8) gives

$$\dot{\nabla}^{\perp}_{\underset{1}{\xi}}\alpha(\underset{1}{\xi},\underset{1}{\xi}) = 0. \tag{2.12}$$

By (2) of the lemma we know that $\alpha(\underset{1}{\xi},\underset{1}{\xi})$ is equal to the mean curvature vector η_x along the curve C. (2.12) means that

$$\dot{\nabla}^{\perp}_{\underset{1}{\xi}}\eta = 0. \tag{2.13}$$

Since x and $\underset{1}{\xi}$ are arbitrary we have shown that the mean curvature vector η of $W_n(g,T)$ is parallel. Thus $W_n(g,T)$ is a sphere.

Conversely, assume that $W_n(g,T)$ is a sphere in $\bar{W}_m(\bar{g},\bar{T})$ and the curve C is a circle in $W_n(g,T)$. In this case the differential equation (2.2) is satisfied.

Since $W_n(g,T)$ is umbilical, we have

$$\alpha(\underset{1}{\xi},\underset{1}{\xi}) = g(\underset{1}{\xi},\underset{1}{\xi})\eta_x = \eta_x. \tag{2.14}$$

From (2.4) and (2.3) we get

$$g(\dot{\bar{\nabla}}_{\underset{1}{\xi}}\underset{1}{\xi},\dot{\bar{\nabla}}_{\underset{1}{\xi}}\underset{1}{\xi}) = g(\dot{\nabla}_{\underset{1}{\xi}}\underset{1}{\xi},\dot{\nabla}_{\underset{1}{\xi}}\underset{1}{\xi}) + H^2 \tag{2.15}$$

where $H = \| \eta_x \|$ is the mean curvature.

In (2.6) we have

$$\alpha(\underset{1}{\xi},\dot{\nabla}_{\underset{1}{\xi}}\underset{1}{\xi}) = g(\underset{1}{\xi},\dot{\nabla}_{\underset{1}{\xi}}\underset{1}{\xi})\eta_x = 0$$

$$A_{\alpha(\underset{1}{\xi},\underset{1}{\xi})}\underset{1}{\xi} = A_{\eta_x}\underset{1}{\xi} = H^2\underset{1}{\xi}.$$

Since η is parallel we obtain

$$\dot{\nabla}^{\perp}_{\underset{1}{\xi}}\alpha(\underset{1}{\xi},\underset{1}{\xi}) = \dot{\nabla}^{\perp}_{\underset{1}{\xi}}\eta_x = 0.$$

Thus (2.5) reduces to

$$\dot{\bar{\nabla}}^2_{\underset{1}{\xi}} \underset{1}{\xi} = \dot{\nabla}^2_{\underset{1}{\xi}} \underset{1}{\xi} - H^2 \underset{1}{\xi}. \tag{2.16}$$

The equation (2.3) is satisfied as a consequence of (2.2), (2.15) and (2.16). Thus the curve C is a circle in $\bar{W}_m(\bar{g}, \bar{T})$. □

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