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Parallel and Minimal Surfaces in Heisenberg Space[1]

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Abstract. After a brief historical review on minimal surfaces in Euclidean spaces $\mathbb{E}^3$ and in Heisenberg spaces H_3, $\mathbb{R}^3(-3)$, the Beltrami formula and parallel surfaces are investigated in these Heisenberg spaces respectively.

Keywords: Heisenberg spaces, minimal surfaces, parallel surfaces

1. Minimal surface in $\mathbb{E}^3$

The generalization from the straight lines, as 1-dimensional objects, being the lines of shortest length in the Euclidean plane $\mathbb{E}^2$, to 2-dimensional objects, i.e. surfaces, in the 3-dimensional Euclidean space $\mathbb{E}^3$, yields the notion of the so-called "minimal" surfaces. More precisely, a surface M in $\mathbb{E}^3$ is called *minimal* when locally each point on the surface has a neighborhood which is the surface of least area with respect to its boundary [12], i.e. when M satisfies the problem of J. A. F. Plateau [25]. The study of the minimal graphs as surfaces in $\mathbb{E}^3$ was historically one of the first applications of the variational problem for double integrals [25]. As a result, J. L. Lagrange in 1760 obtained the following non-linear elliptic partial differential equation: A surface as graph of a function $z = f(x, y)$ is minimal if and only if

$$f_{xx}[1 + (f_y)^2] - 2f_{xy}f_xf_y + f_{yy}[1 + (f_x)^2] = 0,$$

where $f_x = \frac{\partial f}{\partial x}$, $f_y = \frac{\partial f}{\partial y}$ and $f_{xx} = \frac{\partial^2 f}{\partial x^2}$, $f_{yy} = \frac{\partial^2 f}{\partial y^2}$, $f_{xy} = \frac{\partial^2 f}{\partial x \partial y}$.

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J. B. Meusnier in 1775 showed that the condition of minimality of a surface in $\mathbb{E}^3$ is equivalent with the vanishing of its mean curvature function, $H = 0$.

Examples. The first non-trivial examples of minimal surfaces in $\mathbb{E}^3$, – i.e. non-planar minimal surfaces – were found by L. Euler in 1744, considering the class of surfaces of revolution. As a result, he found that the only (non-trivial) minimal surfaces of revolution are the catenoids. Euler called the catenoid as *alysseid.* Around 1775 Meusnier found a second class of non-trivial minimal surfaces, namely the helicoids, and E. Catalan in 1842 proved that the helicoids are the only non-trivial minimal ruled surfaces. H. F. Scherk in 1835 found other examples, the so-called minimal translation surfaces of Scherk, as graph of functions $z(x, y) = g(x) + h(y)$ with separated variables.

2. Minimal surfaces in H_3

In non-Euclidean spaces, in particular in Heisenberg spaces the problem of minimal surfaces was recently studied in [2], [3], [10], [27], [15], for surfaces of revolution with constant mean curvature or Gaussian curvature are studied by R. Caddeo, P. Piu and A. Ratto [7], [8] and P. Tomter [24].

First I would like to mention different metrics that will be used in this paper. We shall denote $\mathbb{R}^3$ endowed with the following metrics:

1. $ds_1^2 = dx^2 + dy^2 + (dz + \frac{y}{2}dx - \frac{x}{2}dy)^2$,
2. $ds_2^2 = dx^2 + dy^2 + (dz + xdy)^2$,
3. $ds_3^2 = \frac{1}{4}(dx^2 + dy^2) + (dz - ydx)^2$

by H_3, $\mathbb{R}^3(-\frac{3}{4})$, and $\mathbb{R}^3(-3)$, respectively. On the 3-dimensional Riemannian manifold H_3 is called the *Heisenberg* 3*-space.* We can define the Lie group structure on $\mathbb{R}^3$ by a multiplication law as follows:

$$(\bar{x}, \bar{y}, \bar{z}) * (x, y, z) = \left(\bar{x} + x, \bar{y} + y, \bar{z} + z + \frac{y\bar{x}}{2} - \frac{x\bar{y}}{2}\right).$$

Then the metric ds_1^2 is left invariant metric on H_3. We can take the following left invariant orthonormal frame:

$$e_1 = \frac{\partial}{\partial x} - \frac{y}{2} \cdot \frac{\partial}{\partial z}, \quad e_2 = \frac{\partial}{\partial y} + \frac{x}{2} \cdot \frac{\partial}{\partial z}, \quad e_3 = \frac{\partial}{\partial z}.$$

This orthonormal frame satisfies

$$[e_1, e_2] = e_3, \quad [e_1, e_3] = 0, \quad [e_2, e_3] = 0.$$

The sectional curvatures are given by:

$$K(e_1, e_2) = \frac{-3}{4}, \quad K(e_1, e_3) = \frac{1}{4}, \quad K(e_2, e_3) = \frac{1}{4}.$$

The identity component $I_0(H_3, ds_1^2)$ of the isometry group is given by the following ([2], [14], [11]):

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \longrightarrow \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ A & B & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} a \\ b \\ c \end{pmatrix}, \tag{2.1}$$

where θ, a, b, and c are real numbers and

$$A = \frac{1}{2}(a\sin\theta - b\cos\theta), \quad B = \frac{1}{2}(a\cos\theta + b\sin\theta).$$

The identity component $I_0(H_3, ds_1^2)$ of the isometry group contains the rotations $(\theta, 0, 0, 0)$ of $\mathbb{R}^3$ around the z-axis and the left translations (0,a,b,c). The Lie group $I_0(H_3, ds_1^2)$ is the semi-direct product of $SO(2)$ and H_3, $SO(2)\alpha H_3$. In [14], the authors define an imbedding of H_3 into the real general linear group $GL(3;\mathbb{R})$; $i : H_3 \to GL(3;\mathbb{R})$;

$$i(x,y,z) = \begin{pmatrix} 1 & x & t \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix},$$

where t is defined by $t(x,y,z) = z + \frac{1}{2}xy$. With respect to this imbedding, the left invariant metric ds_1^2 is expressed as $dx^2 + dy^2 + (dz - \frac{1}{2}xdy)^2$. It easy to see that i is an injective Lie group homomorphism. Hence the Heisenberg group H_3 is identified with the following closed Lie subgroup N of $GL(3;\mathbb{R})$ via i:

$$N = \left\{ \begin{pmatrix} 1 & x & t \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \middle| \; x, y, t \in \mathbb{R} \right\}.$$

The Lie algebra $\mathbf{n}$ is naturally identified with the tangent space of N at $e = (0,0,0)$;

$$T_eN = \left\{ \begin{pmatrix} 0 & x & u \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} \middle| \; x, y, u \in \mathbb{R} \right\}.$$

The exponential mapping $\exp : T_eN \to N$ is given explicitly

$$\exp \begin{pmatrix} 0 & u & w \\ 0 & 0 & v \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & u & w + \frac{1}{2}uv \\ 0 & 1 & v \\ 0 & 0 & 1 \end{pmatrix}.$$

In particular the exponential mapping is a diffeomorphism onto N. See also [11]. The left invariant orthonormal frame e_1, e_2, e_3 corresponds to the basis A_1, A_2, A_3 of T_eN given by

$$A_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; A_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \; A_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

In 1991, Bekkar [2] found the equation of minimal surfaces as graphs of functions $z = f(x,y)$:

$$f_{xx}[1 + (f_y - \frac{x}{2})^2] - 2f_{xy}(f_y - \frac{x}{2})(f_x + \frac{y}{2}) + f_{yy}[1 + (f_x + \frac{y}{2})^2] = 0.$$

In 1992, M. Bekkar and T. Sari [3] classified the ruled surfaces in H_3.

Theorem 2.1. *The minimal ruled surfaces of H_3 ruled by straight lines are (up to isometry of H_3) parts of:*

1. *planes;*
2. *hyperbolic paraboloids $z = \frac{xy}{2}$;*
3. *helicoids;*
4. *surfaces defined by the equation:*

$$z = \frac{\lambda}{2}\left[x\sqrt{(1+x^2)} + \log\left(x + \sqrt{(1+x^2)}\right)\right] - \frac{xy}{2}, \quad \lambda \in \mathbb{R}\setminus\{0\}$$

5. *surfaces with parametrisation:*

$$(x(t,s), y(t,s), z(t,s)) = \left(t + su(t), s, a(t) - \frac{st}{2}\right)$$

where u and a are solutions of the system:

$$(1+u^2+t^2)\ddot{u} - (1+2\dot{u}\dot{a})t\dot{u} = 0,$$
$$(1+u^2+t^2)\ddot{a} - (1+2\dot{u}\dot{a})(t\dot{a} - u) = 0.$$

Remark 2.2. [3] The parts of helicoids are ruled only by geodesic straight lines, but parts of planes and parts of hyperbolic paraboloids are possibly ruled by straight line geodesics and by straight lines which are not necessary geodesics.

Example 2.3. The surface as graph of the function

$$z = \frac{1}{2}\left[x\sqrt{(1+x^2)} + \log\left(x + \sqrt{(1+x^2)}\right)\right] - \frac{xy}{2}$$

is a minimal surface in H_3. This surface is ruled by straight lines L_t spanned by the vectors $(0, 1, -t)$ passing through the points

$$(t, 0, \frac{1}{2}\left[x\sqrt{(1+x^2)} + \log\left(x + \sqrt{(1+x^2)}\right)\right]),$$

which are not geodesics.

Remark 2.4. [15, 13] Let G be a Lie group with left invariant metric and X, Y be two vectors in the Lie algebra $\mathbf{g}$ of G such that $|X| = |Y|$ and $X \perp Y$. If the metric of G is biinvariant then the mapping $\mathbf{r} : \mathbb{R}^2 \to G$ defined by

$$\mathbf{r}(u, v) = \exp(uX)\ \exp(vY)$$

is a flat minimal surface in G. If we choose $G = \mathbb{E}^3$ and $X = (1, 0, 0)$ and $Y = (0, 1, 0)$, then this surface is the xy-plane in $\mathbb{E}^3$. If the metric of G is only left invariant the mapping $\mathbf{r}$ is not necessarily immersion or harmonic map. In the case $G = H_3$, if we choose $X = A_1$ and $e_2 = A_2$ or $e_1 = A_2$ and $e_2 = A_1$, then

$$\mathbf{r}_+(x, y) = \exp(xA_1) * \exp(yA_2),$$

and

$$\mathbf{r}_-(x,y) = \exp(xA_2) * \exp(yA_1),$$

are (nonflat) minimal surfaces in H_3. The minimal surfaces $\mathbf{r}_\pm$ are explicitly given by $\mathbf{r}_+(x,y) = +\frac{1}{2}xy$ and $\mathbf{r}_-(x,y) = -\frac{1}{2}xy$. These surfaces $\mathbf{r}_+$ and $\mathbf{r}_-$ may be considered as $*$-analogues in H_3 of xy-plane in $\mathbb{E}^3$.

Example 2.5. (Translation surfaces [15]) It is well-known that only the minimal translation surfaces in $\mathbf{E}^3$ are planes or Scherk's minimal surfaces. In H_3, we can consider $*$-translation minimal surfaces of nonparametric form:

$$\mathbf{r}(x,y) = (x,0,u(x)) * (0,y,v(y)) = (x,y,u(x)+v(y)+\frac{xy}{2}).$$

The minimal surface equation for the function $f(x,y) = u(x)+v(y)+\frac{xy}{2}$ is

$$u_{xx}(1+v_y^2) - (u_x+y)v_y + v_{yy}\{1+(u_x+y)^2\} = 0. \tag{2.2}$$

We shall solve this differential equation under the assumption that either u or v is constant.

(1) $v(y)$ =constant:

In this case (2.2) becomes $u_{xx} = 0$. Hence we get

$$z = \frac{xy}{2} + ax + b, \ \ a,b \in \mathbb{R} \tag{2.3}$$

(2) $u(x)$ =constant:

In this case (2.2) becomes

$$\frac{d^2v}{dy^2}(1+y^2) - \frac{dv}{dy}y = 0. \tag{2.4}$$

Solving this equation we get

$$z = \frac{xy}{2} + c\left\{y\sqrt{1+y^2} + \log\left(y+\sqrt{1+y^2}\right)\right\} + d, \quad c,d \in \mathbb{R}. \tag{2.5}$$

The minimal surfaces defined by (2.3) and (2.5) are rewritten in the following forms:

$$\mathbf{r}(x,y) = (x,0,u(x)) * (0,y,0) = (0,y,0) * (x,0,u(x)), u(x) = ax+b. \tag{2.6}$$

$$\begin{aligned}\mathbf{r}(x,y) &= (x,0,0) * (0,y,v(y)) = (0,y,v(y)) * (x,0,0)v(y)\\ &= c\left\{y\sqrt{1+y^2} + \log\left(y+\sqrt{1+y^2}\right)\right\} + d, \quad c,d \in \mathbb{R}\end{aligned} \tag{2.7}$$

These formulae imply that the minimal surface (2.6) (resp.(2.7)) is a cylinder over a curve in the xz- (resp. yz-) plane.

Proposition 2.6. [15]

(1) *Let* $\mathbf{r}(x,y)$ *be a cylinder over a curve* $(x,0,u(x))$ *in the* xz*-plane. Then* $\mathbf{r}$ *is minimal if and only if*

$$u(x) = ax + b, \ \ a, b \in \mathbb{R}$$

(2) *Let* $\mathbf{r}(x,y)$ *be a cylinder over a curve* $(0,y,v(y))$ *in the* yz*-plane. Then* $\mathbf{r}$ *is minimal if and only if*

$$v(y) = c\left\{y\sqrt{1+y^2} + \log\left(y + \sqrt{1+y^2}\right)\right\} + d, \quad c, d \in \mathbb{R}$$

(3) *Let* $(x(u), y(u))$ *be a curve in the* xy*-plane, parameterized by the arclength* u*. Then the cylinder over* $(x(u), y(u))$ *is*

$$\mathbf{r}(u,v) = (x(u), y(u), v).$$

Its first and second fundamental forms are

$$I = \left(1 + \omega(\mathbf{r}_u)^2\right) du^2 + 2\omega(\mathbf{r}_u) du dv + dv^2$$

$$II = \{\ddot{x}(u)\dot{y}(u) - \dot{x}(u)\ddot{y}(u) + \omega(\mathbf{r}_u)\} du^2 + du dv,$$

$$\mathbf{n} = \dot{y}(u)e_1 - \dot{x}(u)e_2.$$

The mean curvature H *of the cylinder is*

$$H = \frac{1}{2}\{\ddot{x}(u)\dot{y}(u) - \dot{x}(u)\ddot{y}(u)\} = -\frac{\kappa(u)}{2}.$$

Here κ *is the curvature of the curve* $(x(u), y(u))$.

One can easily check that the base curve of $\mathbf{r}$ is a line in the xy-plane. Hence $\mathbf{r}$ is a plane parallel to the z-axis [20]. (More generally constant mean curvature cylinder is a circular cylinder. Kokubu [17] obtained corresponding results for $SL(2,\mathbb{R})$. Sanini characterised these cylinders in terms of the harmonicity of their tangential Gauss maps. See Propsition 3 in [20].

Remark 2.7. [10] For a surface with parametrization $\mathbf{r}(u,v) = (x(u,v), y(u,v), z(u,v))$ is minimal if and only if satisfy the equation

$$(r_1G - 2s_1F + t_1E)\delta_{23} + (r_2G - 2s_2F + t_2E)\delta_{31} + (r_3G - 2s_3F + t_3E)\delta_{12} = 0;$$

where

$$\delta_{12} = (p_1q_2 - p_2q_1), \quad \delta_{23} = p_2q_3 - p_3q_2, \quad \delta_{31} = p_3q_1 - p_1q_3,$$

$$x_u = \frac{\partial x}{\partial u} = p_1, \quad x_v = \frac{\partial x}{\partial v} = q_1, \ x_{uu} = \frac{\partial^2 x}{\partial u^2 = r_1}, x_{vv} = \frac{\partial^2 x}{\partial v^2 = t_1}, \quad x_{uv} = \frac{\partial^2 x}{\partial u \partial v} = s_1;$$

for p_2, q_2, r_2, s_2, t_2 and p_3, q_3, r_3, s_3, t_3, we change x by y and z respectively.

Theorem 2.8. [10] *Let S be a surface in Heisenberg space with paramerization* $\mathbf{r}(u,v) = (x(u,v), y(u,v), z(u,v))$ *such that*

$$\begin{aligned} x(u,v) &= \alpha_1 u^2 + \beta_1 v^2 + 2\lambda_1 uv, \\ y(u,v) &= \alpha_2 u^2 + \beta_2 v^2 + 2\lambda_2 uv, \\ z(u,v) &= \alpha_3 u^2 + \beta_3 v^2 + 2\lambda_3 uv. \end{aligned}$$

If S satisfies $(\delta_{12})^2 + (\delta_{23})^2 + (\delta_{31})^2 > 0$ and one of the following conditions:

a) $(\delta_{12})(\delta_{23})(\delta_{31}) = 0$,

b) *two colons of the determinant $\delta = det(\alpha, \beta, \lambda)$ are proportional, where $\alpha_i, \beta_i, \lambda_i \in \mathbb{R}$, $\alpha^t = (\alpha_1, \alpha_2, \alpha_3)$, $\beta^t = (\beta_1, beta_2, beta_3)$, $\lambda^t = (\lambda_1, \lambda_2, \lambda_3)$.*

Then the surface S is regular and minimal.

3. Minimal surfaces in $\mathbb{R}^3(-3)$

We consider $\mathbb{R}^3$ with coordinates (x, y, z) and its usual contact form $\eta = \frac{1}{2}(dz - ydx)$, the metric ds_3^2, the tensor field φ and the caracteristic vector field ξ given by

$$\varphi = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & y & 0 \end{pmatrix}, \qquad \xi = 2\frac{\partial}{\partial z}.$$

$\mathbb{R}^3, ds_3^2, \eta, \xi$ is a Sasakian manifold. The vector field $e_1 = 2\frac{\partial}{\partial y}$, $e_2 = 2(\frac{\partial}{\partial x} + y\frac{\partial}{\partial z})$ and $\xi = 2\frac{\partial}{\partial z}$ is an orthonormal basis called *φ-basis*. The sectional curvature of any plane section spanned by a vector X and φX is equal -3, for this reason the Sasakian space $\mathbb{R}^3, ds_3^2, \eta, \xi$ is called Sasakian space form denoted by $\mathbb{R}^3(-3)$ see [4]). G. Zafindratafa in 1997 obtained other examples of Sasakian space form $\mathbb{R}^3(-3)$ as following:

The graphs of the functions $z = axy + P(x) + Q(y)$, where $P, Q \in \mathbb{R}[X]$, are minimal if and only if

1. $z = ax + b$,
2. $z = xy + ay + b$,
3. $z = \frac{1}{2}xy + ax + by + c$,
4. $z = \frac{1}{2}xy + \frac{\varepsilon}{4}x^2 - \frac{\varepsilon}{4}y^2 + \varepsilon ax + ay + b, \quad a, b \in \mathbb{R}, \varepsilon = \pm 1$,
5. $z = \frac{1+\varepsilon\sqrt{1-16a^2}}{2}xy + ax^2 - ay^2 + b, \quad \varepsilon = \pm 1, b \in \mathbb{R}, a \in \left[-\frac{1}{4}, \frac{1}{4}\right]$.

4. Beltrami formula in $\mathbb{R}^3(-\frac{3}{4})$

Let M be a surface of a Euclidean space $\mathbb{E}^3$, with x as its position vector field in $\mathbb{E}^3$. Then the relation between the Laplacian of x and the mean curvature vector field is given by the following *Bertrami formula*:

$$\triangle x = -2\overrightarrow{H}$$

where $\triangle$ is the Laplacian of the surface and $\overrightarrow{H}$ is the mean curvature vector field of M.

A Beltrami formula in $\mathbb{R}^3(\frac{-3}{4})$ is given by the following:

$$\triangle x = -(2H + H\xi_3 + N_3)E_3 - (H\xi_2 + N_2)E_2 - (H\xi_1 + N_1)E_1,$$

where E_1, E_2, E_3 are adapted frame and F_i^j are coefficients of E_i with respect to the orthonormal basis e_j of $\mathbb{R}^3(\frac{-3}{4})$.

$$N_1 = -F_1^3 C - F_2^3 F_3^3, \quad N_2 = -F_2^3 C + F_1^3 F_3^3,$$
$$N_3 = -\tfrac{1}{2} g(E_3, x) - F_3^3 C + F_3^3 \tfrac{x^3}{4},$$
$$C = F_1^3 g(E_1, x) + F_2^3 g(E_2, x) - x^3(\tfrac{1}{4} + (F_1^3)^2 + (F_2^3)^2,$$
$$\xi_1 = (F_1^2 F_3^3 + F_1^3 F_3^2)x^1 - (F_1^1 F_3^3 + F_1^3 F_3^1)x^2 + (-F_1^1 F_3^2 + F_1^2 F_3^1)x^3,$$
$$\xi_2 = (F_2^2 F_3^3 + F_2^3 F_3^2)x^1 - (F_2^1 F_3^3 + F_2^3 F_3^1)x^2 + (-F_2^1 F_3^2 + F_2^2 F_3^1)x^3$$
$$\xi_3 = 2F_3^2 F_3^3)x^1 - 2F_3^1 F_3^3)x^2.$$

An application of the Bertrami formula

The minimal surfaces as graphs of functions $z = f(x, y)$ in $\mathbb{R}^3(\frac{-3}{4})$ are not always harmonic:

$$\triangle x \neq 0.$$

Remark 4.1. J. Inoguchi et al. [15] and G. Zafindratafa [27] studied harmonic maps in N^3 and vector valued harmonic functions on $\mathbb{R}^3(-3)$. A vector valued harmonic function on M into $\mathbb{R}^3(-3)$ is a smooth map φ from M to $\mathbb{R}^3(-3)$ such that $\triangle\varphi = 0$. Here $\triangle$ is the Laplacian of M. A harmonic map φ from M into $\mathbb{R}^3(-3)$ is a smooth map whose tension field vanishes. For the definition of tension field, see [15].

Definition 4.2. *A submanifold M of a Riemannian manifold N is called parallel when $(\bar{\nabla}_X h)(Y, Z) = 0$ where h is the second fundamental form and*

$$(\bar{\nabla}_X h)(Y, Z) = (\nabla_X^{\perp} h)(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z)$$

where $\bar{\nabla}$, ∇, $\nabla^{\perp}$ are the connection of van der Waerden-Bortolotti, the Levi-Civita connection of M, the normal connection of M in N respectively.

In 1948 V. F. Kagan [16] investigated the parallel surfaces in Euclidean space $\mathbb{E}^3$. U. Simon and A. Weinstein [21] generalized Kagan's work to hypersurfaces in $\mathbb{E}^n$ in 1969. J. Vilms [26] studied parallel general submanifolds in Euclidean spaces in 1972. D. Ferus [9] and W. Strübing [22] after showing that the parallel submanifolds of Euclidean space $\mathbb{R}^n$ are related to their extrinsic symmetry, classified them. E. Backes and H. Reckziegel [1] and M. Takeushi [23] investigated parallel submanifolds in spaces of constant curvature. G. Pitis [19] studied parallel submanifolds in the Sasakian space form of dimension $2n+1$, $n \geq 2$. D. E. Blair and C. Baikoussis [5, 6] studied C-parallel submanifolds, i.e., submanifolds such that $(\bar{\nabla} h \parallel \xi)$ in Sasakian space forms. We refer the reader to Ü. Lumiste's paper [18]. We classify parallel surfaces in the Heisenberg 3-space H_3. Before that we recall some definitions and properties of surfaces in Heisenberg space H_3.

Theorem 4.3. [16] *A surface of the Euclidean space* $\mathbb{E}^3$ *is parallel if and only if it is (a part of) a plane* R^2*, a sphere* S^2 *or a round cylinder* $S^1 \times R^1$.

Theorem 4.4. [20] *The Heisenberg group* H_3 *does not admit totally umbilical surfaces, in particularly totally geodesic ones.*

Our main theorem is the following:

Theorem 4.5. *The only parallel surfaces in the Heisenberg group* H_3 *are the vertical planes (planes parallel to the* z*-axis of revolution of* H_3*).* (See Proposition 2.6.)

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