ELEMENTARY SCHOOL STUDENTS' USE OF MATHEMATICALLY-BASED AND PRACTICALLY-BASED EXPLANATIONS: THE CASE OF MULTIPLICATION

Esther Levenson, Dina Tirosh, & Pessia Tsamir

Tel-Aviv University

This paper focuses on elementary school students' use of mathematically-based (MB) and practically-based (PB) explanations. The mathematical context used in this study is multiplication. Two issues are discussed. The first issue is a comparison between MB and PB explanations used by students before they are formally introduced to multiplication in school as opposed to the explanations they use afterward. The second issue is a comparison of the types of explanations used for multiplication without zero as opposed to explanations used for multiplication with zero. Results show that more students use MB explanations than PB explanations. However, when multiplying with zero, many students use another type of justification (ie. rule-based explanation).

It is a long held belief that when elementary school children seek to describe their mathematical thinking or explore mathematical concepts they will use tangible items to manipulate or relate these concepts to real life contexts (e.g., Cramer & Henry, 2002; Fischbein, 1987; National Council of Teachers of Mathematics [NCTM], 1989; NCTM, 2000). This goes along with Piagetian theory which places students of this age at the concrete operational stage. Yet, according to the Standards for School Mathematics (NCTM, 2000), by the "middle and high grades, explanations should become more mathematically rigorous" (p. 61). Fischbein (1987) agreed and took this one step further, "One has to start, as early as possible, preparing the child for understanding the formal meaning and the formal content of the concepts taught" (p. 208). Is it possible to introduce elementary school students to formal mathematics if they are so reliant on concrete examples? Perhaps elementary school students are too young for rigorous explanations but not too young for explanations that are less formal but nevertheless rely solely on mathematical notions. This study focuses on the types of explanations that elementary school students use. By focusing on the types of explanations used we reexamine the premise that elementary school students need explanations that rely on tangible items or real life stories and investigate the possibility of introducing explanations that rely solely on mathematical notions in these grades.

Explanations have been classified in many different ways throughout the years. This study investigates mathematically-based (MB) explanations and practically-based (PB) explanations. MB explanations employ only mathematical notions. PB explanations use daily contexts and/or manipulatives to "give meaning" to mathematical expressions (Koren, in press). This classification distinguishes between explanations that are based solely on mathematical notions but are not necessarily rigorous, and complete, formal explanations. Formal explanations are usually referred to at the high school and undergraduate level. The term PB explanation was coined to include any explanation that does not rely solely on mathematical notions.

Much research has been done relating to the use of PB explanations in the elementary school mathematics classroom. Many of these studies (e.g., Koirala, 1999; Nyabanyaba, 1999; Szendrei, 1996; Wu, 1999) found that each type of PB explanation has its own set of pitfalls which need to be avoided or remedied by the teacher. Other studies have investigated the use of mathematical explanations in the elementary school that do not rely on manipulatives or real life stories (eg., Ball & Bass, 2000; Lampert, 1990). However, these studies took place in inquiry-based classrooms where mathematical discourse was encouraged. The current study investigates how students who study in more traditional classrooms explain multiplication without zero and with zero.

METHODOLOGY

Subjects

Subjects in this study were divided into younger and older students. The first group consisted of twenty second graders from three different schools who had not yet been exposed to multiplication in class. The second group consisted of ninety-one third, fifth, and sixth grade students from four different schools who already had experience with multiplication in school.

Instruments

Because young children express themselves better orally it was decided to interview the second grade students as opposed to using questionnaires. During the interview, students were asked to solve and give their own explanations for multiplication problems without and with zero. Older students were asked to fill out questionnaires. Both instruments included the following multiplication problems:

3 x 2 =

2 x 3 =

3 x 0 =

0 x 3 =

The first two problems sought to establish how multiplication without zero was solved and explained and if the subject used the commutative property of multiplication as an explanation. The second two problems allowed us to investigate how subjects solved and explained multiplication of a non-zero number by zero. Specifically, these two questions investigated if the types of explanations used by the subject for multiplication by non-zero numbers differed from the explanations used for multiplication with zero. Furthermore, these two questions allowed us to investigate if the subject differentiated between 3x0 and 0x3. The problem 3x0 fits well into the definition of multiplication as repeated addition when the multiplier is a positive integer and indicates the number of times 0 is to be added to itself. However, when the multiplier is non-positive, as in the case of 0x3, difficulties may arise. Therefore, it was of particular interest to investigate how subjects would explain this problem and how prevalent the use of the commutative property would be in this case.

Procedure

As stated before, second graders were each interviewed individually. Every interview was audio taped. Older students filled out questionnaires in their classroom with their class teacher and the researcher present. Students were told to solve each problem and provide explanations that they might use if they were asked to explain their solution to a friend in class who did not know the answer. Students worked individually without consulting the teacher, the researcher, or other students.

RESULTS

This section discusses the results of the interviews and questionnaires. First, we discuss the various explanations and how they were categorized into MB and PB explanations. Explanations for multiplication without zero and with zero were categorized in a similar matter and examples are given for each. We then present separately the distribution of the types of explanations used for multiplication without zero and with zero and discuss these results. Finally, we look at the differences between explanations used by second grade students who had not yet been introduced to multiplication in class and third, fifth, and sixth grade students who already had experience with multiplication in school.

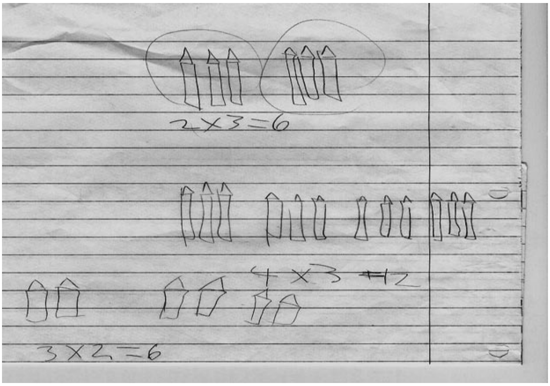
Categorization of explanations

MB explanations

As stated above, MB explanations employ only mathematical notions. In this category we included explanations that did not rely on the use of pictures, concrete objects, or stories. Many explanations were based on the definition of multiplication as repeated addition. As one fifth grader stated, "Multiplication is pretty simple. It's like quick addition". This type of explanation usually involved representing the multiplicand (the second number) the number of times that is indicated by the multiplier (the first number) and then successively adding these numbers. An example of this is the following explanation for 3x2 given by a second grader, "2 and another 2 is 4 and another 2 is 6". Similarly, when explaining 3x0, one fifth grader wrote, "0+0+0=0". Also in this category were explanations that relied on sequencing, such as one second grader's explanation, "I just counted by 2's, … 2, 4, 6". There were also students who based their explanation on the word "times", such as, "This one is like saying 3 two times, which would be six". A similar explanation was given by a second grader for 0x3, "You don't have to write any times 3". Finally, explanations that were based on the commutative property of multiplication were considered MB explanations. Included in this category were explanations that explicitly used this property calling it by its proper name as well as explanations that stated that the order of the factors is irrelevant to the solution. As one sixth grader wrote, "2x3=6. It is the exact same as before, (3x2). In multiplication just like addition it does not matter which order the numbers are".

PB explanations

PB explanations were defined above as explanations that use daily contexts and/or manipulatives to "give meaning" to mathematical expressions. Most explanations in this category included pictures and stories that the students used to "give meaning" to the multiplication task. The following is an illustration of a second grader's PB explanation of why 2x3=6 and 3x2=6:



This student originally answered that 2x3 equals 12. When explaining her solution, she drew 2 sets with 3 pencils in each. The student realized her mistake and then wanted to figure out how many sets of 3 pencils she would need in order to have 12 pencils. This led her to draw 4 sets of 3 pencils and write "4x3=12". Finally, she drew 3 sets of 2 pencils to illustrate why 3x2=6.

One fifth grader drew 3 circles with 2 x's in each as an explanation for 2x3=6. When explaining 3x0 he drew 3 empty circles and wrote "3 groups of nothing". Another fifth grader wrote a story, "You have three guests and each of your guests wants two pancakes. How many do you have to make?" When explaining multiplication with zero, a different fifth grader wrote, "You have 3 ice creams but you don't eat any. How many did you eat? (0)". One sixth grader used a picture accompanied by a story. He drew 3 large circles with 2 dots in each and wrote underneath, “There are 3 cages and 2 animals are in each cage. Now count all the animals that are in the cages”.

Rule-based (RB) explanations

RB explanations have not been mentioned earlier because they were not the focus of this study. However, when explaining multiplication with zero many students sited the rule that every number times zero must equal zero. This explanation was not categorized as either MB or PB, and was therefore given its own category. For an explanation to be categorized as RB it had to be clear that the student was generalizing his explanation to all multiplication examples with zero as a factor. An example of this can be seen from the third grader who wrote, “Everything that you multiply by zero is equal to zero”.

Other

Included in this category are students who did not offer any explanation at all for their answers and students who offered explanations that were not clearly MB, PB, or RB.

Multiplication without zero

In this section we discuss the students’ explanations for 3x2=6 and 2x3=6. It should be noted that most students used the same type of explanation for both examples. In other words, most students who used a MB explanation for 3x2 used a MB explanation for 2x3 and likewise for PB explanations. That being said, three students (about 3%) gave different types of explanations for the two different examples. A few students gave both MB and PB explanations for the same task. Results are summed up in Table 1. Percentages are based on the number of students in each grade.

Table 1: Distribution of the types of explanations per grade for multiplication without zero (in %)

Task	3 x 2				2 x 3			
Grade	2	3	5	6	2	3	5	6
	n = 20	n = 35	n = 32	n = 24	n = 20	n = 35	n = 32	n = 24
MB	85	86	56	63	85	89	72	71
PB	10	-	19	29	10	-	19	21
MB & PB	5	-	22	-	5	-	6	-
Other	-	14	3	8	-	11	3	8

Results show that in every grade, students are more likely to use MB explanations than PB explanations. One might have expected that younger children are more likely to base their explanations on their life experiences while older, more mathematically experienced students, would choose MB explanations. However, this was not the case in this study.

Multiplication with zero

Two new issues arose in students' explanations for multiplication with zero that were not present in the examples without zero. First, not all second graders knew the correct results of multiplication with zero. Second, many older students offered rule-based explanations without explaining the rule. In this section we will first discuss the solutions given by younger and older students and then present the types of explanations used for multiplication with zero.

Although all second graders interviewed knew multiplication without zero, 15% incorrectly solved 3x0 and 40% incorrectly solved 0x3 (see Table 2). It should be noted that many children changed their minds several times and only their final answers are considered. It is very interesting to note that all (except for one) of the younger students, who answered incorrectly, claimed that 3 times 0 (or 0 times 3) equals 3 and not all students who answered one question correctly answered the second correctly. These results show that although students may know multiplication without zero, it does not necessarily follow that they will know multiplication with zero. Possibly, this is because children's first experiences are with the set of natural numbers. Students, especially younger ones, often relate zero to "nothing" or "emptiness". At times, this can lead to an incorrect solution.

Table 2: Distribution of the types of explanations per grade for multiplication with zero (in %)*

Task	3 x 0				0 x 3			
Grade	2	3	5	6	2	3	5	6
	n = 20	n = 35	n = 32	n = 24	n = 20	n = 35	n = 32	n = 24
MB	80 (70)	40 (33)	31 (31)	21 (21)	75 (35)	57 (46)	37 (37)	21 (21)
PB	5 (5)	-	16 (16)	16 (16)	10 (10)	-	13 (13)	17 (17)
RB	-	29 (29)	34 (34)	46 (46)	-	23 (23)	31 (31)	37 (37)
MB &PB	5 (-)	-	-	-	-	-	-	-
Other	10 (10)	31 (21)	19 (16)	17 (17)	15 (15)	20 (14)	19 (13)	25 (25)

* Percentages of correct solutions are given in parentheses.

One of the second grade students who claimed that 3x0=3 explained, "Because 3 times 0, you don't add anything so it stays the same number". Another second grader was also confused by the zero:

Interviewer: What about 3 times 0?

Student: It's 3. (This is an automatic response.)

Interviewer: Tell me, why do you think it's 3?

Student: Cause it's 0, so it's nothing.

Previously, this student had drawn a picture using sets of tally marks to illustrate why 3x2=6 and 2x3=6. In light of this drawing, the interviewer asked the student if he could draw a picture for 3x0. He drew 3 tally marks. Then, to illustrate 0x3, he drew a big empty circle and said, "It's just nothing".

Other students answered correctly that 3x0=0 but were still confused as to whether zero should be considered a number or not. This is illustrated by the following exchange with a second grader who had clearly used repeated addition when multiplying 3 by 2 but found multiplying by 0 quite different:

Interviewer: And what is 3 times 0?

Student: 0.

Interviewer: Why?

Student: Because…it doesn't have a number. If you had one, then it could be different. Because you can't do 3 *times* 0. It's still 0.

Interviewer: Why can you do 3x2 but you can't do 3x0?

Student: Because 0 is a number but it's… it's nothing. It's nothing.

Among older students, only 6 third graders (17% of the third graders) answered incorrectly that 3x0=3 and 0x3=3. One fifth grader did not answer at all. Overall, 92% of the older students knew that multiplication with zero always results in zero. This is a great increase over the second graders and shows that almost all students who learn multiplication in class knew that multiplying with zero results in zero.

From Table 2 we see that there are similarities and differences in the results for 3x0 and 0x3. MB explanations are used more often than PB explanations for both tasks. However, RB explanations were used less for 0x3 than for 3x0 as many students used the commutative property, a MB explanation, for the second task. Although one might think that older children, aware of the commutative property, would always have a ready explanation for 0x3, this was not always the case. In fact, a comparison of the tasks among sixth graders shows an increase in the use of "other" explanations for 0x3 than for 3x0. Perhaps this is due to the multiplier being a non-positive integer.

Second graders did not use RB explanations at all. These explanations were only used by students who had been introduced to multiplication in school. However, among the older group of students, it should be noted that for both tasks, the use of RB explanations rises from grade to grade. This does not imply that students are unaware of what lies behind the rule they cited. However, the question remains as to what might have caused this dramatic increase. Is this the only explanation presented in class? If not, then why do so many students recite this rule and not a different explanation?

When comparing the results of multiplication without zero to that of multiplication with zero, we see a decrease in the use of both MB and PB explanations for multiplication with zero, most likely due to the use of RB explanations. Significantly, there are more "other" explanations for multiplication tasks with zero than for without zero. This may be the result of students' difficulties, regardless of age, incorporating zero into the number system.

Discussion

Although the focus of this paper was on MB and PB explanations, the use of RB explanations cannot be ignored. Before students have formal learning they are not exposed to any rules. They try to give meaning to the mathematics, either by connecting it to life experiences or by basing it on already known mathematical concepts. After being introduced to multiplication in class, students give up using

MB explanations in favor of more RB explanations. Is this a trend that we want to encourage? One of our goals as mathematics educators is to help our students move from PB explanations to MB explanations. In the beginning of this paper we asked if it is possible to introduce more formal mathematics to young children. Knowing that the move to formal mathematics may be difficult, we should examine the possibility of introducing more MB explanations to elementary school students.

This study shows that even young students are capable of using explanations that rely solely on mathematical notions. Is this true only for multiplication tasks? We need to examine students' use of MB explanations in other mathematical contexts as well. We also need to investigate how these findings may be used in practice by teachers in the classroom and, in line with Fischbein's (1987) recommendation, investigate how MB explanations may be used to prepare students for the formal content of mathematics.

References

Ball, D. & Bass, H. (2000). Making believe: The collective construction of public mathematical knowledge in the elementary classroom. In D. Phillips (Ed.), *Yearbook of the National Society for the Study of Education, Constructivism in Education*. Chicago, IL: University of Chicago Press.

Cramer, K. & Henry, A. (2002). Using manipulative models to build number sense for addition and fractions. In B. Litwiller (Ed.), *Making sense of fractions, ratios, and proportions* (pp. 41-48). Reston, VA: The National Council of Teachers of Mathematics, Inc.

Fischbein, E. (1987). *Intuition in science and mathematics*. Dordrecht, the Netherlands: Reidel Publishing Company.

Koirala, H. (1999). Teaching mathematics using everyday contexts: What if academic mathematics is lost? In O. Zaslavsky (Ed.), *Proceedings of the 23rd Conference of the International Group for the Psychology of Mathematics Education*, III (pp. 161-168). Haifa, Israel.

Koren, M. (in press). Mathematically-based and practically-based explanations in the elementary school. *Mispar Hazak* (in Hebrew).

Lampert, M. (1990). When the problem is not the question and the solution is not the answer. *American Educational Research Journal,* 27(1), 29-63.

National Council of Teachers of Mathematics (1989). *Curriculum and evaluation standards for school mathematics*. Reston, VA: NCTM.

National Council of Teachers of Mathematics (2000). *Principles and standards for school mathematics*. Reston, VA: NCTM.

Nyabanyaba, T. (1999). Whither relevance? Mathematics teachers' discussion of the us of 'real-life' contexts in school mathematics. *For the Learning of Mathematics,* 19(3), 10-14.

Szendrei, J. (1996). Concrete materials in the classroom. In A. J. Bishop (Eds.), *International Handbook of Mathematics Education*, (pp. 411-434). the Netherlands: Kluwer Academic Publishers.

Wu, H. (1999). Basic skills versus conceptual understanding: A bogus dichotomy. *American Educator*, 23 (3), 14-19, 50-52.