VISUAL REASONING IN COMPUTATIONAL ENVIORNMENT: A CASE OF GRAPH SKETCHING

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Abstract

This paper reports the case of a form six (grade 12) Hong Kong student's exploration of graph sketching in a computational environment. In particular, the student summarized his discovery in the form of two empirical laws. The student was interviewed and the interviewed data were used to map out a possible path of his visual reasoning. Critical features that enable visual reasoning in graphing software are discussed.

Introduction

Visualization is gaining important status in mathematics and mathematics education (Arcavi, 2003; Hershkowitz, Arcavi & Bruckheimer, 2001). With the rapid development of ICT, attempt to use virtual visualization as a mean for learners to construct mathematical knowledge and meanings has been an active research agenda (see for examples, Leung & Lopez-Real, 2002; Noss, Healy & Hoyles, 1997). In Chan (2001), a study was conducted to investigate the impact of graphing software on the learning of curve sketching in a form six (grade 12) class in Hong Kong. Graphmatica and Winplot were chosen as the graphing environments that students could use during the exploration tasks. In particular, one of the tasks was to ask students to investigate (using the computer only) the shapes of the family of graphs that has the symbolic form

$$y = (x+a)^m (x+b)^n (x+c)^p$$

where m, n, p are positive integers and a, b, c are real numbers. A student, Kelvin, came up with two empirical laws that could facilitate the sketching of these graphs without resorting to rigorous mathematical analysis. He claimed that this theorizing was a consequence of what he saw on the computer screen. Furthermore, he was able to apply his laws to sketch the graphs of the form

$$y = (x+a)^{\frac{m}{n}} (x+b)^{\frac{p}{q}} (x+c)^{\frac{r}{s}}.$$

Kelvin was interviewed in depth twice after he presented his findings to the class. In this paper, we will try to interpret on Kelvin's thinking patterns in his exploration and to sieve out possible elements that might contribute to visual reasoning in a computational graphing environment.

The Exploration Task

First, we outline roughly the tasks in the study that we are interested in. It consists of a sequence of explorations using Graphmatica or Winplot.

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A1. Investigate the family of graphs of the form $y = x^n$ where n is a positive integer.

A2. Investigate the family of graphs of the form $y = (x+a)^n$ where n is a positive integer and a is any real number.

A3. Investigate the family of graphs of the form $y = (x+a)^m (x+b)^n (x+c)^p$ where m, n, p are positive integers and a, b, c are real numbers.

B1. Investigate the family of graphs of the form $y = x^{\frac{m}{n}}$ where m and n are positive integers.

B2. Investigate the family of graphs of the form $y = (x+a)^{\frac{m}{n}} (x+b)^{\frac{p}{q}} (x+c)^{\frac{r}{s}}$ where m, n, p, q, r, s are positive integers and a, b, c are real numbers.

There were subtasks for each item and students were asked to work in pairs.

Kelvin's Path of Visual Reasoning

Tasks A1 and A2

Kelvin had learnt how to sketch the graphs $y = x^2$ and $y = x^3$ before this exploration, however he didn't know what $y = x^n$ looked like in general. The computer helped him to discover the general shape of $y = x^n$.

Excerpt from interview 2: (I: the interviewer; K: Kelvin)

I: What did you learn in the tasks concerning $y = x^n$?

K: When the index is odd, the shape looks like:

       When the index is even, the shape looks like:

I: Do you mean that you don't know this beforehand and you learnt it from the computer during the exploration tasks?

K: Yes, but I knew the basic graph $y = x^2$, $y = x^3$ in form four but I do not know the graphs of $y = x^4$, $y = x^5$ before.

At this stage, it appears that when Kelvin saw the shapes of $y = x^n$ on the computer screen for different values of n, he was able to make a conjecture on the general shape of $y = x^n$. His prior knowledge on $y = x^2$ and $y = x^3$ played a pivoting role in this process of generalization since these were the two graphs that Kelvin had an analytical understanding on, and this knowledge on the polarity of the index of x seemed to *visually agree* with the general case. It's interesting to note that Kelvin ascribed the word "basic" to the graphs of the two specific cases. It seemed that the visual experience of $y = x^n$ was integrating with what Kelvin had already known. A process of meaning exchange between symbolic representation and computer generated visual image was taken place: the polarities of the index of x became shapes.

<u>Task A3</u>

When Kelvin was asked to present his findings for exploration Task A3, he proposed two laws, law of combination and law of continuity, which he claimed would facilitate the sketching of the graph of $y = (x+a)^m (x+b)^n (x+c)^p$ in general. A summary of these two laws is as follow:

Law of Combination for $y = (x+a)^m (x+b)^n (x+c)^p$

(A)   (i)   When the sum of the indices of the factors is odd, then the overall shape of the curve would look like:

Positive leading coefficient   Negative leading coefficient

   (ii)   When the sum of indices of factors is even, then the overall shape of the curve would look like an U ($\cup$) or an inverted U ($\cap$) with some 'mountains' and 'valleys':

Positive leading coefficient   Negative leading coefficient

(B)   The factors of the function are considered independently. The graphs of the factors are drawn individually first, then they are combined by reflecting (when necessary) both ends of the graphs about x-axis, with reference to the signs of y when $x \Diamond \pm \infty$.

Law of Continuity

During the combination of separate curves of the factors, continuity of curve should hold.

During the interview, Kelvin indicated that part (A) of the law of combination was inspired by his discovery in Tasks A1 and A2.

Excerpt from interview 2: (I: the interviewer; K: Kelvin)

I: Kelvin, how do you think out the Law A(1) and A(2)?
K: This originates from a lesson earlier but I forget which lesson. Is it the first lesson, I am not sure?
I: What is the focus of that lesson?
K: The lesson we studied the $y = x^2$, $y = x^3$ and mm…
I: Is it the lesson we studied $y = x^n$?
K: Yes.

For Part (B) of the law of combination, Kelvin credited its formulation as a consequence of his interaction with the computer and his interpretation of the computer's visual feedback. Here's how he described the process of the discovery.

Excerpt from interview 2: (I: the interviewer; K: Kelvin)

I: There is another question I like to ask. How did you think of the method of combining the graphs of all separate factors to get the overall graph?

K: This is owing to the display of curve (*on the computer screen*) during the lesson. For instance, for the graph of $y = (x-1)^2(x+2)^4$, the software shows (*he made the following sketch*):
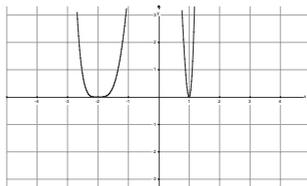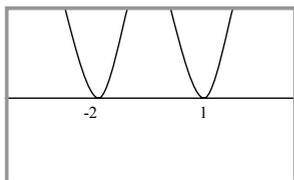
Kelvin's sketch of what he saw in computer

Actual image from the software

K: But, after applying function of 'zoom out', the separated curves are 'joined together' like the graph below.
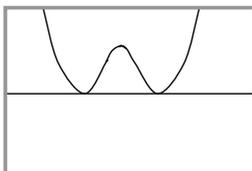
Kelvin's sketch of what he saw on the computer of the graph after adjustment of the y-scale

K: So, I thought that I have to draw the separate curves of the factors first and then joined them together, and reflections of separate curves about x-axis may be needed.

The immediacy of visually displaying the scaling effect (via the zoom function) that the graphing software could provide for Kelvin was the key to open the door to this deep speculation. Kelvin was experiencing visually a process of discovering global features (the whole) via local properties (the parts), in particular, of how (prompted by the visual feedbacks) to paste together local characteristics to reveal a "generalized whole". More interesting, the *inability* of the graphing software to show the whole continuous graph at the first instance when Kelvin put in the equation was probably the most significant moment in the whole discovery process. Kelvin *saw* the graphs (separated from each other) that he was acquainted with in Task A. This laid

the foundation for his two laws. Then the 'zoom out' function of the software that "joined together" the "separated curves", as described by Kelvin, gave Kelvin a visual fact (the continuity of the graph) that he needed to accept and to cope with. This opened up a cognitive space in which he had to simultaneously fuse together all the visual information and his prior knowledge to formulate an explanation, even laws. This was how Kelvin explained why he needed the law of continuity.

Excerpt from interview 2: (I: the interviewer; K: Kelvin)

K: In form six, we were taught the concept of continuity and differentiability. These curves should be continuous.

I: Yes, but what stimulates you, does the computer help you in any way?

K: After drawing the curves of separate curves. They have to be continuous. Therefore, the curves are smooth and continuous. And it is observed that some of the graphs of factors are reflected about x-axis to maintain the characteristic of continuity. Therefore, I try to isolate this important feature in additional to the law of combination.

I: Do you know why these curves are continuous?

K: Naturally equations of these forms should be continuous … this is my guess.

The zoom function acted as a channel to a global view of variation, allowing Kelvin to experience contrast, separation, generalization and fusion. It is interesting to compare the zoom function in a graphing software with the drag function in a dynamic geometry environment, both of them serve as a kind of interactive amplification tool for the learner to see global behavior via variation (see Leung, 2003). Kelvin's explanation on how to paste two separated curves together was quite intriguing.

Excerpt from interview 1: (I: the interviewer; K: Kelvin)

K: (Described with drawing) In cases when the two intercepts of two curves are closed, the shape results in a small 'valley' (*he probably meant 'low hill'*) like this:

   But when the two intercepts are separated far away, say +100 and −100, y value of the joining curve (⌒) between this two basic curves becomes very large.

I: Yes. Do you know the reason?

K: I think that when the factors are considered independently, the curves themselves have their own shapes and sizes. But, when they are combined, they have to suit each other. For instance, in order for these two curves to be combined, their joining parts have to be diminished a bit. Suppose the two shapes are identical, it is unreasonable to squeeze the jointed part. They do not work like this. So, the part will be cut off.

The following graphs were probably typical of what Kelvin saw on the computer screen that led him to this line of reasoning.

$$y = (x-1)^2(x+1)^4 \qquad\qquad y = (x-2)^2(x+1)^4$$
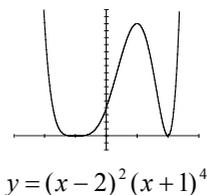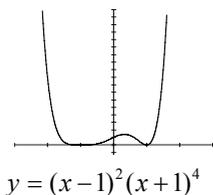
It is worth noting that Kelvin used phrases like "the curves themselves have their own shapes and sizes" and "they have to suit each other" to describe the combination process of the two graphs. The visual variation resulting from symbolic variation somehow endowed "personalities" to the graphs and Kelvin reacted to them by "feeling".

Excerpt from interview 1: (I: the interviewer; K: Kelvin)

I: Your reasons are full of 'feeling'!

K: Yes, my explanation is full of art. The explanation of Fan is of logical.

I: What do you mean by 'art'?

K: … See the shapes of graphs so as to determine how to draw. But I seldom … . This differs from approach adopted by Fan who looks (considers) numbers and he did series of process of 'arithmetic' and then draw graph. I draw graphs directly and then observe (find features). So, he is logical while I am of art.

(Fan was another student who approached these tasks in an analytical way)

Very seldom do we explicitly relate doing mathematics with doing art. However, in this graph sketching exploration via a computational environment, Kelvin 'explained' Mathematics like an artist interpreting a piece of visual artwork. "See the shapes of graphs so as to determine how to draw." This is in contrast with the usual way of graph sketching, that is, determine how to draw (via rigorous analysis), then see the shapes. The graphing software situated Kelvin in a medium in which he could "see" mathematics in ways that enabled him to "feel" mathematics. Could this element of "feeling" via "seeing" (whatever is its meaning at this point) be a relevant factor that can contribute to visual reasoning?

Tasks B1 and B2

After Task A and the formulation of the two laws, Kelvin went ahead to attempt Task B. He had no difficulty in Task B1 to discover the basic shapes of $y = x^{\frac{m}{n}}$. To sketch the graph of $y = (x+a)^{\frac{m}{n}}(x+b)^{\frac{p}{q}}(x+c)^{\frac{r}{s}}$, Kelvin used the two laws he developed and illustrated the sketching procedure with a specific example.

Excerpt from interview 1: (I: the interviewer; K: Kelvin)

K: To answer this logically, let use an example to explain. (Kelvin drew the graphs on paper.)
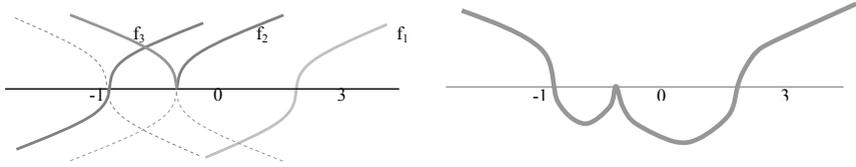
Discrete curves drawn by Kelvin   A continuous curve drawn by Kelvin

$$y = (x+1)^{\frac{1}{3}} x^{\frac{2}{3}} (x-3)^{\frac{1}{3}}$$

K: The latest theory is that for x > 3, the graph follows the curve with x-intercept 3 (f_1). In the middle, graph follows the middle one (f_2) while on the left graph follows the left one (f_3). In order to make the curve continuous, the basic graphs have to be reflected about x-axis.

I: Why?

K: This is my observation. And I haven't work out any proof.

I: You haven't got any reason but you think that this is correct. Are you sure that this method is correct? Are you sure?

K: Observation only …

I: Okay.

The actual graph of $y = (x+1)^{\frac{1}{3}} x^{\frac{2}{3}} (x-3)^{\frac{1}{3}}$ looks like:
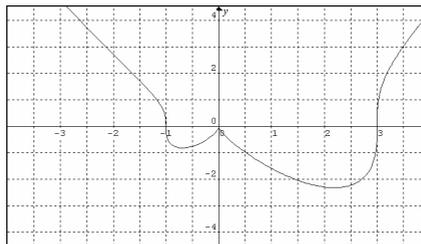
We can see the striking resemblance between Kelvin's sketch and the actual graph. Kelvin was able to apply his laws in a different situation. This may imply that these two laws were rich in mathematical content even though Kelvin hadn't "work out any proof" and his reasoning relied very much on intuition and observation. In this respect, as far as exploring graph sketching using the computer only, Kelvin's discovery has to be an exemplary one.

Final Remark

We have tried to trace a possible path on which Kelvin developed his two empirical laws for graph sketching using graphing software environment. The zooming (re-

scaling) function of this environment seems to play a pivotal role in the visual reasoning process. Local scales give partial pictures that can evoke prior knowledge while global scales reveal the whole that might create cognitive conflicts which need to be resolved, hence the drive to make conjecture and to theorize. The ease of transition between local images and the global images of a graph via the zoom function turns the awareness of the learner to focus on the critical features that determine the continuity (or discontinuity) between local properties and global properties. These variable visual images can further stimulate a learner's intuitive feeling about mathematical behaviours that might contribute significantly in the visual reasoning process. Kelvin's two laws were heuristic and they are operationally quite effective. Mathematical analysis needs to be performed to verify and to fine-tune them. Nevertheless, these laws were indigenous to the graphing computational environment and they were obtained by systematic visual observation via scaling. There are at least two critical features in graphing software that are conducive to visual reasoning. The first one is its ability to instantly let the learner sees the graph of practically any equation. In this sense, there is no need to sketch the graph, rather, the question now becomes why certain graph looks certain way. Secondly, the amplification of the ability to vary visual images in different modalities (in our case, re-scaling) allows a learner to see invariant properties (e.g., Kelvin's two laws) of the "generic graph". These two features are the catalysts in a visual reasoning process. We hope this paper serves as a convincing example for them.

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