STUDENTS BUILDING ISOMORPHISMS

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This paper reports on five students' explorations of structural relationships between problem situations that they worked on over several years as participants in a long-term study. In particular, we describe the case of students who recognized isomorphisms between and among two problem situations and who used particular features of the problems to explain Pascal's Identity.

INTRODUCTION

According to Greer and Harel (1998), an important component of mathematical cognition is recognizing structural relationships between situations and distinguishing these from surface features. They give as examples the mathematical thinking of Poincaré and a fourth-grade student, Brandon, who spontaneously recognized the mathematical equivalence between two problems he had been exploring. The distinguished mathematician and the fourth grader, they point out, recognized the common structure between two situations that on the surface appear different:

> These are two examples of mathematicians – the first a great French genius, the second a fourth grade New Jersey student – experiencing insights about structural identity underlying what, on the surface, appear to be different situations (Greer and Harel, 1998, p. 6)

For further details on Brandon's work, see Maher (1998).

THEORETICAL FRAMEWORK

According to Davis and Maher (1997), in order to build accurate mental representations for mathematical situations, students need to draw on fundamental basic ideas that they have built and recognize how these ideas are related. This involves a form of reasoning in which learners map corresponding relational structures from one situation to another. Davis indicates that thinking about new ideas involves making connections to ideas that were already present, and in so doing, one often makes use of metaphors in order to communicate these ideas. But Davis points out:

> Metaphors are far more important than mere tools of communication – they form a large part of the mental representations by which we think (Davis, 1990, p. I.4).

In describing the importance of metaphors in building new conceptualizations, Davis suggests that having a collection of basic metaphors provides one with a powerful way of looking at new mathematical phenomena. Using the interpretative framework of Davis' work on metaphoric thinking, we trace the mathematical reasoning and the building of isomorphism of six students who did

mathematics together over several years as participants in a long-term longitudinal study. During the course of the study, the students worked on several combinatorics problems as part of a counting strand. Their work led to investigations of various combinatorial theorems, including Pascal's Identity, the addition rule for Pascal's Triangle.

The questions that guide our analysis are: (1) What metaphors did students use in their investigation of Pascal's Triangle? (2) How did they use the metaphors to explain Pascal's Identity? (3) In what ways were the metaphors used to provide a convincing justification for Pascal's Identity?

PASCAL'S TRIANGLE

Pascal's Triangle shows the number of ways to generate combinations of objects. The number of combinations of r objects selected from n objects is given by the rth entry of the nth row of Pascal's Triangle, often written as $C(n,r)$:

$$C(n,r) = \binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

Pascal's Identity is given by the equation $C(n,r) = C(n-1,r-1) + C(n-1,r)$.

Metaphors for Pascal's Triangle

The numbers in Pascal's Triangle are often used to find the coefficients of the binomial expansion: $C(n,r)$ gives the coefficient of the rth term of $(x+y)^n$. But these numbers can also be used to investigate some combinatorial problems. The following two problems are isomorphic to each other and to the problem of finding the coefficients of $(x+y)^n$. The students investigated versions of these problems, starting in fourth grade and continuing through high school. They form the basis of the metaphors the students used to talk about Pascal's Triangle.

1. **The Pizza Problem:** $C(n,r)$ gives the number of possible pizzas with exactly r toppings when there are n toppings to choose from. The students worked on the 4-topping ($n=4$) and 5-topping ($n=5$) version of this problem in 1993, during fifth grade. They investigated other forms of the problem during high school.

2. **The Towers Problem**: $C(n,r)$ gives the number of towers (built from Unifix cubes and with two colors to choose from) that are n cubes tall containing exactly r cubes of one of the colors. The students first encountered the 4-tall ($n=4$) version of this problem in third grade. They worked on it again in fifth grade and revisited it in high school.

METHODOLOGY

Data for this analysis is taken from after-school problem-investigation sessions during the students' last three years of high school and from individual follow-up task-based interviews. All sessions were videotaped, usually with two cameras. All

student work was preserved. Summaries were made of all videotaped sessions, and they were coded for critical events related to student use of metaphors and discussion of isomorphic relationships. All critical events were transcribed, and transcripts were reviewed for accuracy.

The five sessions discussed here took place on February 6, 1998, March 1, 1999, May 12, 1999, April 3, 2002, and July 22, 2002.

In the 1998 session, a group of students discussed the pizza and towers problems and the binomial expansion. The students were asked to explain how the 6th row of Pascal's Triangle is related to the 6-tall towers problem and the 6-topping pizza problem.

In the March 1999 session, a group of students explored Pascal's Triangle and Pascal's Identity in terms of the towers and pizza problems. They were asked to think about how entries in Pascal's Triangle are related to the towers and pizza problems.

In the May 1999 session, a group of students was asked to discuss the meaning of combinatorial notation and to explain the addition rule in terms of that notation. They were asked to write the general form of Pascal's Identity.

The two sessions in 2002 were individual-student task-based interviews; in both cases, students were asked to explore the meaning of the numbers in Pascal's Triangle and to discuss Pascal's Identity. Michael was interviewed in April, and Romina was interviewed in July.

RESULTS

Throughout the longitudinal study, the students visited and revisited several problems in combinatorics, including the two listed above. During their high school years, they were encouraged to investigate number patterns that recurred in those problems. In the course of their discussions and with the guidance of researcher questions about how the numbers are related, they made sense of Pascal's Triangle and Pascal's Identity in terms of these isomorphic problems. They made use of metaphors they already knew to extend and generalize their knowledge and build new ideas. We describe here some instances where the students used metaphors in order to explain the numbers in Pascal's Triangle and to give meaning to Pascal's Identity.

The Case of Ankur

In 1998, Ankur and Jeff were asked to explain how Row 6 of Pascal's Triangle (1 6 15 20 15 6 1) can be connected to the towers and pizza problems:

Ankur: So there's one with no red. There's six with one red. There's 15 with two reds.

Jeff: ...Two reds, 20...

Jeff: ...With 3 reds.

Ankur: 15 with four reds.

Jeff: ...With 4 reds.

Ankur:	Six with five reds.
Jeff:	And one with no…
Ankur:	No, six reds.
Jeff:	One with six reds.

Later in that session, the students were asked to explain how the two middle numbers in Row 5 of Pascal's Triangle (10 and 10) are added to produce the middle number in Row 6 of Pascal's Triangle (20). Here, Ankur explained how $C(6,3)=C(5,2)+C(5,3)$.

R2:	Hm. the first 10 in that row of 5 high has 2 reds and 3 blues. We're counting reds.
Ankur:	Yeah.
R2:	And the second 10 has...
Ankur:	Three reds and two blues.
R2:	Three reds and two blues. Now coming down here, the 20 is supposed to count the ones that have...
Ankur:	Three reds and three blues.
R2:	Right. So how do the two 10s add to give the 20?
Ankur:	Because in these 10, when there's three reds and two blues, you want to make it three reds and three blues. So you put a blue on top of each one. …
R1:	… To preserve the three reds.

The Case of Stephanie

In the March 1999 session, Stephanie was a member of a group of students exploring the pizza problem and its relationship to Pascal's Triangle. Stephanie related the numbers in Row 3 of Pascal's Triangle to pizzas with three toppings to choose from and the numbers in Row 4 to pizzas with four toppings to choose from. She explained two instances of Pascal's Identity, as shown in Figure 1. First she explained how 1+3=4 in terms of building one-topping pizzas in Row 4 from zero- and one-topping pizzas in Row 3:

OK, this, to get four pizzas with one topping, you already have three pizzas with one topping. And the plain pizza becomes the pizza with the new topping. … OK, so this becomes, instead of one plain pizza, this is one pizza with one topping. Cause this one's getting like the pepperoni thrown into it.

Then Stephanie used the metaphor of building two-topping pizzas in Row 4 from one- and two-topping pizzas in Row 3 to explain 3+3=6:

…So then here, um, you have six pizzas with two toppings. Now you already have three pizzas with two toppings. So these three pizzas with one topping get an extra topping added on. … So these become three pizzas with two toppings. And then three pizzas with two toppings plus three pizzas with two toppings equal six pizzas.
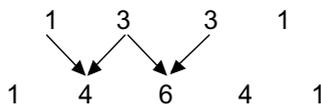
Figure 1: Adding to Create the Next Entry in Pascal's Triangle

The Case of Jeff

In the May 1999 session, Michael and Jeff explored the same instance of Pascal's Identity that Stephanie had described, also using the metaphor of adding toppings to pizzas.

Michael: So all these 3s would either move up a step onto the next category and, uh, have two toppings or they might stay behind and still only have one if they have the zero. So three, three… [Michael draws lines as shown in Figure 3.] Well, three get a topping, go to this one [Michael points to the 6], and three won't, will stay. [Michael points to the 4.] … So now this guy's going to have, without toppings. [Michael points to first 1 of 1 3 3 1 row.] You're going to add a topping on to him. That's going to be one topping. These three [points to first 3 of 1 3 3 1 row] with one topping won't get one so, you know, you can put them in the same category as this one.

Jeff: That's their four? Yeah.

Michael: That's four.

Jeff: Those are your four.

Michael: And you know, the three that had two toppings [the second 3 in the 1331 row] won't get any. And you could put them in together with the ones that did get something. [Michael points to the lines going from the two 3s to the 6.] That's why you would add.

Later, Brian joined the group, and Jeff explained the general form of Pascal's Identity to him, which was written in the form shown in Figure 2.

Jeff: We, we were explaining why you add.

Brian: All right, keep going.

Jeff: And why do… because when you add another topping like onto it, this one… Say the toppings were one and zero. [Jeff points to N choose X.]

Brian: Uh huh.

Jeff: If it gets a topping, that's why it goes up to the X+1. [Jeff points to N+1 choose X+1.]. And since it doesn't get anything it'll stay the same. And in this one [Jeff points to N choose X+1], it's staying the same, right? [Jeff looks at Michael.]

Michael: Yeah.

Jeff: And that's why it's going there. Like saying that's the zero. [Jeff writes a small zero at the bottom of N choose X+1.]

Brian: Okay.

Jeff: You're going to there. Make sense?

Brian: Yes. It actually does.

Jeff: So, so that would be the general addition rule in this case.

$$\binom{N}{X} + \binom{N}{X+1} = \binom{N+1}{X+1}$$

Figure 2: Pascal's Identity in Students' Notation

The Case of Michael

In the April 2002 interview, Michael was asked to discuss Pascal's Identity. First he explained what the numbers in Pascal's Triangle meant in terms of the pizza metaphor, starting with Row 2 (1 2 1):

> If you had no toppings, that would be one pizza. … If you're having only, using just one topping, you can make two possible pizzas with that. And then if you have all, all the toppings, that's one. Right. And then automatically you, I see that, that relates to this row. [Michael points to his paper, where he has written 1 2 1.] And I'm pretty sure it would go down, this is like a third topping, and a fourth topping. Now I think the way I, um, thought about it is, like, the row on the outside would be your plain pizza. And there's only one way to make a plain pizza. And the next, you know, from then on, the next one over would be how many pizzas you can make, um, using only one topping, and then so on until you get to the last row which is, um, all your toppings. And, once again, you can only make one pizza out of that. …

Then Michael used the pizza metaphor to explain Pascal's Identity. He equates moving to the left (adding a number to the number on its left) with not adding a topping and moving to the right (adding a number to the number on its right) with adding a topping:

> Well, you're starting off with this, you know, this group of pizzas that has no toppings. And this group of pizzas that has one. So when you, when you go up, you have the choice of adding one more. … to that one that had nothing, you could either not give it that extra topping. … Or you can. So for those pizzas that you do give that extra topping, it moves to the right. And for the others it moves to the left. And that's kind of why it doubles. The amount doubles each time.

The Case of Romina

In the July 2002 interview, Romina was shown a diagram of Pascal's Triangle in combinatorial form and asked how the addition rule worked. Romina pointed to Row 1, which contained *C(1,0)* and *C(1,1)*, and explained the connection between those numbers and the pizza and towers problems. Romina's explanation paralleled Ankur and Jeff's explanation from 1998.

I think this is, how many, how many toppings, like the top number, like the one choose one or one choose zero would be how many toppings. Or I mean if we were talking about towers. … It would be like how many high and this [pointing to Row 1] would be like with zero reds, with one red. This [pointing to Row 2, containing *C(2,0)*, *C(2,1)*, and *C(2,2)*] would be with two high with zero reds, one red, two reds. And it just keeps going like [pointing to Row 3] three high, zero reds, one red, two, then three reds. So it would be like three high and like out of those, you choose how many blocks of each color.

Romina was asked if she could explain the addition rule in terms of towers. She responded with a similar explanation to that offered in 1998 by Ankur:

So...OK, I think, the way it goes, it's like this two [the 2 in the middle of Row 2: 1 2 1], you can take, you could be either direction with it. You're either going to add another, add another red, say, so it goes to two red, or you're going to add another blue, so it stays one red.

Later Romina was asked to explain in terms of pizzas. Her response echoed portions of the discussion from 1999, but she went further and explained the isomorphism between the pizza problem and the towers problem.

R3:   Um. Can you look at this in terms of pizzas too? The way Michael does?

Romina: … if you have two toppings, if you have a possibility of two toppings, in this one [first entry of Row 1 of Pascal's Triangle], you don't have any toppings. And this one [second entry of Row 1], you have one topping.

R3:   OK, so you can pick from two toppings, but you don't put any on.

Romina: Yeah, you don't have to necessarily put it. You have two toppings to pick from. And then, what he [Michael] did with this one is either you could...now, you could add a third toppings to your pizza. … Like you have three options, you could either not add anything to the pizza. Or you could just add one more topping.

R3:   All right. So when you said, "add one more topping," or "not add one more topping," how do you... can you relate that to red and blue?

Romina: You either, it's either, like, you add one more red block, or you just keep it consistent and add, just add another blue. ... So blue would be like nothing, like not an ingredient, and red would be an ingredient.

CONCLUSION AND IMPLICATIONS

These students built connections between some of the problems they worked on throughout their school years (pizza and towers) because they recognized their corresponding structures and their relationships. Using the metaphors and their understanding of the isomorphic relationships as a way of looking at Pascal's Triangle, these students were able to describe and explain Pascal's Identity. They made use of metaphors they already knew to extend and generalize their knowledge and to build new ideas.

Over many sessions students provided lucid explanations of what the numbers in Pascal's Triangle mean. They made use of two isomorphic problems to provide a convincing justification of Pascal's Identity. Their knowledge appears deep and durable. This suggests that giving students rich problems throughout their school career can help them build a collection of basic metaphors that they can later use when building new, more formal, mathematical ideas.

Note

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