SPREADSHEET GENERALISING AND PAPER AND PENCIL GENERALISING

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The Purposeful Algebraic Activity Project[1] is a longitudinal study of the development of pupils' algebraic activity in the early years of their secondary schooling. Analysis of data from a spreadsheet-based teaching programme and from semi-structured interviews leads us to identify three features of the spreadsheet environment that appear to shape pupils' generalising: focus on calculations; use of notation; and feedback. We discuss how pupils' experience of generating spreadsheet formulae can potentially support pupils' generalising in a paper and pencil environment.

BACKGROUND

Generalising falls within Kieran's (1996) definition of generational activity. Several researchers have reported the difficulties that pupils meet when generating expressions and equations (for example Clement, Lochhead and Monk, 1981). There is some evidence that computer programming environments can support pupils in formalising their generalisations and in developing an understanding of variables (Noss, 1986; Hoyles and Sutherland, 1989; Tall and Thomas, 1991). The improvement in formalising has been attributed to the procedural nature of computer programming and to the use of a symbolic language.

Researchers have suggested that spreadsheets can support pupils in developing an understanding of variables. In a longitudinal study of two groups of 10-11 year old pupils working on traditional problems, Sutherland and Rojano (1993) conclude that 'a spreadsheet helps pupils explore, express and formalise their informal ideas' (p.380), moving from thinking with the specific to symbolising a general rule. Rich examples of pupils exploring, expressing and formalising their ideas offer some insight into how the spreadsheet shapes their activity (Sutherland and Rojano, 1993; Ainley, 1996; Friedlander, 1999).

Ainley (1996) analyses the generational activity of two pupils working on the task, Sheep Pen, which challenges pupils to find how to make a rectangular sheep pen with the largest possible area using 30 metres of fencing set against a wall. The description of the work of the pupils shows how they formalised their generic method of calculating the length of the sheep pen from any width. The pupils made sustained attempts write a spreadsheet formula, and with researcher intervention at the latter stage of their activity, they successfully generated the formula '=30-B11*2'. Ainley identifies the purposeful nature of the task as central to the success of the pair, who

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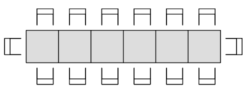
had not been introduced to formal algebraic notation. Friedlander (1999) discusses the changes that pupils make to their formulae within a classroom rather than research setting, alongside the reasons for these changes. These are identified as ‘unreasonable computer output, peer discussion or intervention by a neighbour or by the teacher’ (p.339). Friedlander reports that although pupils saw the spreadsheet as a natural tool in moving from the particular to the general, they also experienced cognitive difficulties. For example, pupils frequently generated large quantities of data without questioning whether the data was reasonable.

Sutherland (1995) found that low achieving 14-15 year olds, who had worked on a unit which required them to write an algebraic version of a spreadsheet formula were able to use their knowledge in a paper and pencil test. It is suggested that the ‘the spreadsheet symbol and the algebra symbol came to represent “any number” for the pupils’ (p.285).

The Purposeful Algebraic Activity project aims to explore the potential of spreadsheets as tools in the introduction to algebra and algebraic thinking. To this end we have designed and implemented a spreadsheet-based teaching programme with five classes of pupils in Year 7 (aged 11-12), their first year at secondary school. Pupils across the attainment range were represented in the five classes, set by ability, from two secondary schools. The teaching programme consisted of six tasks, amounting to approximately 12 hours of activity over the course of a year. The pupils’ usual teachers, with whom we have worked throughout the project, taught all of the lessons. We are also conducting regular semi-structured interviews with two cohorts of pupils (those participating in the teaching programme, and those following their usual algebra curriculum). The interviews are designed to cover a number of themes, including generational activity. We have reported our initial findings from analysing pupils’ responses to the ‘tables and chairs’ question (Ainley, Wilson and Bills, 2003).

Tables and chairs are arranged like this in the school dining room:



Can you explain how to work out how many chairs are needed for each long table?

Can you write it down?

Figure 1: The ‘tables and chairs’ question

We distinguished between pupils generalising the context (‘you need three for the end tables and two for the rest of the tables’) and generalising the calculation (‘you’d double it then add the two on the end’). In drawing out implications for the design and implementation of the teaching programme, we suggested that tasks could signal the need to describe a calculation, and that teachers could encourage pupils to articulate and generalise their calculations. This paper arises from our analysis of how the spreadsheet can shape pupils’ generalising in such tasks.

DATA COLLECTION

In each of the teaching programme lessons, a range of data was collected. A pair of pupils was video taped whilst working on the task, and screen recording software was used in most lessons. These sources of data were collated into a narrative of the pupils' work. Each narrative includes transcribed dialogue interwoven with details of pupils' non-verbal behaviour and interaction with the computer. In addition, audio recordings were made of teachers' interactions with pupils using a radio microphone; these were semi-transcribed. Field notes were made by the first named author, and examples of written work and spreadsheet files were collected. In our analysis of the data from the teaching programme we have used NVivo software to code examples of pupils engaging in various aspects of generational, transformational and global meta level activity (Kieran, 1996).

We also refer here to data from interviews with pairs of Year 7 pupils prior to and following the teaching programme. The first named author conducted all of the interviews, with each question presented in written form and read aloud. Pupils were encouraged to discuss their responses, and all of the interviews were video and audio taped. Transcripts were annotated to include non-verbal behaviour and any written work. Some interviews were also conducted with high attaining pupils at the beginning of Year 8. These pupils had participated in the teaching programme whilst in Year 7. These interviews were conducted as part of a related study, which aims to identify how pupils mobilise spreadsheet-based knowledge during algebra tasks, and to explore teaching strategies which support this. The interviews followed a similar pattern and included the 'tables and chairs' question, but the pupils were also offered the spreadsheet if they had difficulty formalising their generalisation on paper.

PAPER AND PENCIL GENERALISING

Prior to the teaching programme, pupils' responses to the 'tables and chairs' question show evidence of algebraic activity, as well as some specific difficulties in generating a symbolic version of the rule. A middle attaining pupil generalised the calculation as 'times it by two and then add two on it,' writing '6x2=12' and '12+2=14.' When asked if she thought that someone would understand what to do for a longer table she explained 'the same like, say if it was ten then ten times two is twenty and then you add two.' When pressed to write something with letters she suggested '6tx2=12' and '12+2=14c', describing the original calculation using letters as objects. We have other examples of pupils using letters in the same way after the teaching programme e.g. '10tx2=20t+2c' and '10t=22c'. These pupils typically understood the calculation, and were able to explain how someone should interpret what they had written. In our interviews with pupils following their usual curriculum we have also seen examples of pupils' written expressions matching their verbal articulation:

> 'one table equals two chairs and then you'd have to add two chairs to either end' 1t=2c+2

We have observed that pupils' written generalisations, and any changes they make, tend to reflect their verbal attempts to work with the relationship.

SPREADSHEET GENERALISING

The following vignette of Jason and Beatrice's response to the 'tables and chairs' question comes from a Year 8 interview. When asked how to tell the caretaker to calculate how many chairs she would need to get out of the store room, Beatrice and Jason both generalised the context, but in different ways:

Beatrice	I'd do it by having two per table and then like adding two at the end
Jason	… For every table you need two chairs … with the exception of the two end ones … when you add three

Jason and Beatrice knew that sometimes a formula can be written but Jason felt that it was not possible for this question. It appears that he was generalising the context rather than the calculation. The 'two per table' way of looking at the arrangement was revisited, but they were unable to write a symbolic version. In our teaching programme, Jason and Beatrice experienced entering spreadsheet formulae, often to generate data to solve a problem. Although this question focused on formalising the generalisation rather than solving a purposeful problem, there was time at the end of the interview to go back to the question and offer the pupils the spreadsheet below.

	A	B
1	number of tables	number of chairs
2		

Jason and Beatrice were asked to put in a formula that would work out the number of chairs. After initial hesitation, because some time had passed since they had used spreadsheets, Beatrice remembered that they should use A2 because the computer wouldn't understand 't'. Having clarified that they should try and write a formula that would work for any number of tables, Beatrice generalised the calculation:

Beatrice	Times two plus two
Jason	(pointing to chairs on diagram) Fourteen, six, double it plus two …
Researcher	So now you've got to teach the computer. You know how to do it but you've got to teach the computer how to do it
Jason	Try um, number of tables, (in B2 Beatrice types '=A1') um
Beatrice	Times two
Jason	Times two plus two (…) ~ try it, enter (Beatrice enters '=A1*2+2' which gives '#VALUE!') (.) No. What have we done wrong? (..)

They decided to 'try it with six,' entering '6' in cell A2 and correcting the formula to '=A2*2+2.' They pressed enter and both smiled when the number 14 appeared. They had a clear understanding of the generalised rule:

Researcher	What do you understand by A2? If someone said what, what's A2, what does it mean, how would you explain that?
Jason	Erm, any number … but in this case it's "six 'cause it's in that column
Beatrice	'Cause it's in that cell

Using the spreadsheet, Jason and Beatrice formalised the generalisation. In our analysis of the range of data from the teaching programme, we have coded three features of working in a spreadsheet environment which we identify as valuable in supporting pupils' generational activity: focus on calculations; use of notation; and feedback. Reflections on the interplay between the teachers' pedagogic practice and pupils' construction of meaning have also clustered around these three features.

Focus on calculations

Using the spreadsheet, Jason and Beatrice moved from generalising the context ('two per table and then like adding two at the end') to generalising the calculation in natural language ('times two plus two') to formalising that calculation on the spreadsheet ('=A2*2+2'). In a paper and pencil environment, pupils can 'read' their written generalisations in an idiosyncratic way. But on the spreadsheet, the activity of writing a formula necessitates expressing the calculation in a formalised way.

In our analysis of the teaching programme data, we have coded pupils' use of arithmetic examples to support them in writing a spreadsheet formula. We found that pupils can successfully move from expressing a calculation for a particular number to writing a spreadsheet formula. In the Sheep Pen task (with 39m of fencing), we have seen examples of teachers encouraging pupils to articulate their calculations and then move on to teaching the spreadsheet their method. Whether pupils' responses are specific calculations such as 'take eight away from thirty-nine' or include a sense of variable, such as 'you add on *what(.)ever you need* to make thirty-nine which is thirty-one,' scaffolding questions such as 'What sum have you done?' and 'How did you work that out?' were successfully employed with pupils across the ability range.

Use of notation

Jason and Beatrice understood the use of a symbol to represent a variable. After Jason initially suggested using 't', they used A1 (the label 'number of tables') before remembering the need to use A2. A cell reference such as A2 is clearly not an abbreviation for an object. Beatrice was aware that A2 refers to the contents of the cell. But it also takes on another layer of meaning because if the formula is filled down, using A2 enables calculations to be made in the whole column. Perhaps this is what Jason had understood. Within a spreadsheet, the notation conventions need to be adopted in order to drag the formula down or to change the number in cell A2.

We have many examples from the teaching programme of pupils from across the ability range successfully constructing spreadsheet formulae, and using these formulae in ways that clearly suggest that they are thinking about variables rather than the particular number in the cell. The teachers involved in the research have also reinforced the cell reference as a variable e.g. 'We just say whatever number's in that box.' We have also seen examples of pupils using a single letter rather than a cell reference whilst working in a spreadsheet environment. A pupil tried to write a formula using 'w' for the width of the sheep pen: '39-2w*='. In Sheep Pen, one pair of pupils started to use 'A,' the name of the column in their formula.

Judith (teacher) Right, why A? A is all of these in this whole column

Pupil Aren't we meant to do loads of sums, that's why, that's why we put

The fact that pupils use letters themselves suggests that it would be useful to explore further the potential of naming cells and columns, a facility offered by a spreadsheet.

Feedback

The spreadsheet offers immediate numerical feedback, enabling pupils to check their formula. Jason and Beatrice were expecting the number 14 to appear, and when it did they were satisfied that they had written the formula correctly. In the teaching programme, we have coded examples of pupils interpreting feedback from the spreadsheet, often to correct their formula. We have seen a number of pupils achieve success by working through a calculation that gives incorrect feedback. For example, in Sheep Pen, one pair had written the formula '=A8*2-39' (rather than '=39-A8*2') for the length of the sheep pen, giving numerical feedback of –19. But whilst they knew it was incorrect it was not until they were encouraged to work through the formula substituting A8 with 10 (their current width) that the pupils corrected the formula by themselves. This activity of interpreting feedback is fleeting by its nature, with pupils inspecting and changing formulae fluently. It does appear that the teacher can play a useful role in encouraging pupils to attend to whether feedback is reasonable, which is something that some pupils tend to overlook (as in Friedlander, 1999), and also to encourage pupils to consider what the spreadsheet has done.

SPREADSHEETS SUPPORTING PAPER AND PENCIL GENERALISING

The three features of the spreadsheet that we have identified as significant in shaping pupils' generalising are not embedded in a paper and pencil environment. There is no requirement to think in terms of calculations or to use specific notation, and there is no feedback. However, we do have evidence that spreadsheets can support generalising in a paper and pencil environment as indicated by the work of Maria and Jane, high attainers interviewed at the beginning of Year 8. Maria and Jane followed a similar course of action to Jason and Beatrice. Maria initially expressed the relationship in natural language (describing the context):

Maria Would it be like, er, 'cause there's two tables for "each bit of the thing and then there's always gonna be two on each end (pointing throughout)

They then moved directly onto trying to write an equation in standard notation:

Jane You could write um (..) t … for tables (.) plus (…) …

Maria Could put like two c equals one t

Jane Yeah, um

Maria Plus two?

T=tables
C=chairs

$t = 2c + 2c$

When asked what that would look like written down both Maria and Jane wrote 't=2c+2' then wrote another 'c' for chairs next to the 2. Without prompting, Jane checked what she had written for one table and two tables, but her activity reflected her understanding of the context rather than actually substituting the values into the equation that they had written:

Jane	If you had one, "one table then there'd be four chairs which is two plus two … If you had two tables there'd be four chairs plus two which equals six

When asked to try to write a spreadsheet formula, Jane wrote '=A2+2', but realised she had made a mistake when she saw the feedback of 3 chairs for 1 table. She referred to what they had written in standard notation.

Jane	No (laughs) I know what I've done wrong (..) (deletes formula in B2) Erm equals, equals this number (….) this number, what did we write down here?
Maria	(…) I don't ↑think this formula's right (points to 't=2c+2c' written on paper) (..) 'cause it's just like (..) …
Researcher	What makes you think it's not right Maria?
Maria	Just 'cause like, that's just four, it doesn't seem right like …I don't know. 'Cause you can't like really (.) get if you wanted to work out like what would it be with ten tables. You can't really do it for that

Working on the spreadsheet had led to Maria questioning their written generalisation. She quickly generalised the calculation ('times two plus two') and they quickly went on to enter the formula '=A2*2+2'. Having entered numbers for the tables and dragged the formula down, Maria and Jane corrected their written equation:

Both	Number of tables (Jane looks at spreadsheet)	
Jane	Tables times two plus two equals c (writes)	$t \times 2 + 2 = c$
Maria	Yeah (look at what Jane has written)	
Jane	t, t times two plus two equals c. Tables times two plus two equals chairs	

DISCUSSION

Our analysis has highlighted three features of the spreadsheet environment which we see as significant in shaping pupils' generalising. The focus on calculations, use of notation and feedback all act as scaffolds for pupils' formalising, keeping pupils in touch with arithmetic procedures alongside their verbal attempts to work with the relationships. This supports evidence that computer programming environments can help pupils to formalise their generalisations. Moreover, as illustrated in the interview with Maria and Jane, these features can also support pupils' generalising in a paper and pencil environment. Working on the spreadsheet was a sufficient scaffold for these high achieving pupils to correct their written generalisation. Their work on the spreadsheet did not involve complex calculations that they would have been unable to carry out mentally. But embedded into their spreadsheet experience was a need to formalise a calculation using a cell reference as a variable. The class had not been taught to replace cell references with standard algebra notation.

Maria and Jane were actually using the spreadsheet in their interview. The extent to which experience of using spreadsheets might influence pupils' generational activity *when the spreadsheet is not present*, is unclear. The interviews that took place with pupils after the teaching programme (at the end of Year 7) were paper and pencil based. We will use these to try to identify subtle influences from the spreadsheet experience on pupils' responses. This will be a major focus of our future analysis across a range of questions and types of algebraic activity.

We have seen in the teaching programme that aspects of teachers' pedagogic practice can usefully advance the three features we have identified here. In terms of mobilising spreadsheet-based knowledge away from the spreadsheet, we suggest that the teacher has a major role to play. Pedagogic strategies could include: asking pupils to think how they would write a formula if they were using a spreadsheet; experience of naming cells/ columns with letters on a spreadsheet; and using substitution to check a formula. The experiences of learners and teachers working with such strategies will be a focus of our further research.

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