UNIVERSITY STUDENTS' CONCEPTIONS OF FUNCTION

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"Function", as it is understood today, formulates one of the most important concepts of mathematics. Nevertheless, many students do not sufficiently understand the abstract but comprehensive meaning of function and problems concerning its didactical metaphor are often confronted. The present study examines the interpretation of the concept of function among second year students of the Department of Education, at the University of Cyprus, and outlines their misunderstandings and possible obstacles in fully grasping its meaning. Results have shown that students' perception of function appears in isolated components of mathematical ideas associated with the concept of function.

INTRODUCTION

A historical perspective of the way the concept of "function" came to exist in contemporary mathematics would reveal centuries of discussions among mathematicians. On the other end, the didactical metaphor of this concept seems difficult, since it involves three different aspects: the epistemological dimension as expressed in the historical texts; the mathematics teachers' views and beliefs about function; and the didactical dimension which concerns students' knowledge and the restrictions implied by the educational system. On this basis, it seems natural for students of secondary education, in any country, to have difficulties in conceptualizing the notion of function.

The present work examines the interpretation of the concept of function by second year students of the Department of Education, at the University of Cyprus. Since the participants come from different secondary school directions, the present investigation is likely to reveal various types of misunderstandings. Predominantly, these students are prospective primary school teachers, who will in a way transfer their mathematical thinking to their future students.

EPISTEMOLOGICAL DIMENSION AND THE DIDACTICAL METAPHOR OF FUNCTION

The concept of function is central in mathematics and its applications. It emerges from the general inclination of humans to connect two quantities, which is as ancient as Mathematics. Nevertheless, what directed to the idea of managing unique relations, which is accepted in the formal definition of function, was the need for calculations, within the framework of Analysis, especially during modernity. Based on the definitions of Euler, Bernoulli, and Cauchy, Dirichlet in 1837, concluded in the expression *"Variable y is said to be a function of variable x defined in the*

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interval a<x<b, if to every value of variable x in this interval corresponds only one value of variable y, independently from the form of the correspondence". In this definition the concept of variable includes a timeless intelligible election of value within the space of real numbers. The set-theoretic definition of Dedeking was the next stage (Davis J.P., & Hersh R., 1981).

Consequently, to sum up, function, as a typical mathematical concept, is a mental construction that was integrated rather recently in mathematics. It is a matter of synopsis and congregation of different experiences and conceptual tools that mathematicians and scientists initially used to solve problems and assemble theories.

Due to this historical concentration, the notion of function is so abstract that presents many difficulties in its didactical metaphor. Different epistemological approaches that led to the meaning of function through its long historical evolution are disrupting into the teaching guides and textbooks of mathematics in a confusing way. The complexity of this didactical metaphor has been a main concern of mathematics educators and an active question in the research of mathematics education (Dubinsky & Harel, 1992; Sierpinska, 1992; Gomez & Carulla, 2001; Hansson & Grevholm, 2003). Moreover, the understanding of functions does not appear to be easy, because of the diversity of representations associated with this concept, and the difficulties presented in the processes of articulating the appropriate systems of representation involved in problem solving (Yamada, 2000). Therefore, a substantial number of research studies have examined the role of different representations on the understanding and interpretation of functions (Thomas, 2003; Zazkis, Liljedahl, & Gadowsky, 2003).

Researchers usually investigate the epistemological obstacles, on the basis of the historical study of the concept of function, and propose teaching methods, which aim at overcoming these obstacles. In practice, different approaches that are applied in mathematics instruction concerning the concept of function result in exposing to the students the pieces of a puzzle consisting of a vague set of extracted information, that possibly merge at university level in mathematics. Sierpinska (1992) gives a viable example of such an approach supporting that formulae, graphs, diagrams, word descriptions of relationships and verbal expressions, compose an uncertain schema of thoughts.

We believe that further research regarding the understanding and use of functions by university students is needed, so that their difficulties and misconceptions are identified. This could lead to planning and applying appropriate and efficient instruction at university level, for improving students' comprehension about functions. The present study aims to provide answers to the following research questions: a) How do students conceive and use the concept of function? b) How do students recognize functions in multiple representations? It should be noted that the main concern of the present study is beyond the measurement of the success rate to the proposed tasks, and focuses on the connections of students' conceptions about functions, as indicated by their responses to the tasks.

METHOD

The sample of the study consisted of 164 students who attended the course "Contemporary Mathematics" during the first semester of the academic year 2003-2004, at the University of Cyprus. The course is compulsory for the students of the education department, and can be elected by the students of mathematics department students. The questionnaire was completed by 154 second year students of the education department and 10 four year students of the department of mathematics.

Students were asked to complete a written questionnaire that included tasks of recognition of functions among other forms, given in various types of representation (verbal expressions, graphs and mapping diagrams or algebraic expressions). A variety of functions were used: linear, quadratic, discontinuous, piecewise and constant functions. Furthermore, students were asked to provide a definition of what function is and two verbal examples of functions application in real life situations. Below we give a brief description of the questions:

Question 1: Recognition of functions between four given verbal expressions (Q1a, Q1b, Q1c, Q1d).

Question 2: Construction of the characteristic function of a set (Q2).

Question 3: Construction of the algebraic expression of a function given in verbal expression (Q3).

Question 4: Recognition of functions between six given graphs (Q4a, Q4b, Q4c, Q4d, Q4e, Q4d).

Question 5: Construction of a graph from an algebraic expression of one of the functions of the previous question (Q5).

Question 6: Recognition of functions between five given graphs (Q6a, Q6b, Q6c, Q6d, Q6e).

Question 7: Construction of a graph of a function with domain distinct points (Q7).

Question 8: Recognition of functions between four given diagram mappings (Q8a, Q8b, Q8c, Q8d).

Question 9: Definition of function (Q9).

Question 10: Examples of functions from their application in real life situations (Q10).

Correct and wrong answers were accounted for all the questions. Answers to questions 9 and 10 were given additional codes as it is further described.

The definitions given by the students were additionally coded as follows:

D1: *An approximately correct definition*. In this group the following type of answers were included: (i) accurate definition, (ii) correct reference to the relation between variables but without the definition of the domain and range, (iii) definition of a special kind of function (e.g. real function, function one-to-one or on to, continuous function).

D2: *Reference to an ambiguous relation*. Answers that made reference to a relation between variables or elements of sets, or a verbal or symbolic example were included in this group.

D3: *Other answers*. This type of answers made reference to sets but without relation, or reference to relation without sets or elements of sets.
D4: *No answer*

As for the examples, they were coded as follows:
X1a: Example of a function with the *use of discrete elements of sets*.
X1b: Example of a *continuous function* from physics
X2: Example of a *one-to-one function*.
X3: Example presenting an *ambiguous relation* between elements of sets.
X4: Example of an *equation* (verbal or symbolic).
X5: Example presenting an *uncertain transformation* of the real world.
X6: *No example*.

For the analysis and processing of the data, Gras' s implicative statistical analysis was conducted by using the computer software CHIC (Bodin, Coutourier, & Gras, 2000). A similarity diagram, which allows for the arrangement of tasks into groups according to their homogeneity, was produced. The notion of 'supplementary variables' was also employed in the particular analysis. Supplementary variables enable us to explain the reason for which particular groups of variables have been created and indicate which objects are "responsible" for their formation. In our study, secondary school direction and field of study (i.e. education or mathematics) were set as 'supplementary variables'. Consequently, we were able to know which school direction or study field contributed the most to the formation of each group.

RESULTS

The results are presented into three sections. In the first section we present some indicative answers given in the last two questions and in the second section we present the percentages of success. In the third section we present the results of the implicative analysis using software CHIC.

(i) Some indicative answers
We restrict the qualitative analysis to the answers given in the last two questions, since they are of most interest.

In the question requiring the definition of function the answers that gave an approximately correct definition were grouped together. "*Function is a relation between two variables so that one value of x (or the independent variable) corresponds to one value of y (or the dependent variable)*" were accounted in this group. Answers like "*Function is an equation with two depended variables*", "*Function is a relation in which an element x is linked with another element y*" or even "*Function is a mathematical relation connecting two quantities*" were coded as D2. As D3 we have coded answers, which made reference to sets, but did not mention relation, or involved relation and not sets or elements of sets, that is answers like "*Function is relation*" or "*Function is a mathematical concept that is influenced by two variables*" or "*Function is the identification of parts of a set*".

The correct examples of a function were of two kinds (X1a and X1b). Examples of the first kind were "*Each person corresponds to the size of his shoes*", "*Each student corresponds to his/her mark at the test*" made use of sets with discrete elements. The second type of examples presented a continuous function mainly from physics such as "*The height of trees is a function of time*", "*Atmospheric pressure is a function of altitude*". The examples presenting a function one-to-one were coded separately as X2. Such answers were "*Every citizen has his own identity number*", "*Every graduate has his own different degree*", "*Every country corresponds to its own unique name*". As X3 we coded the examples presenting a relation between elements or variables but without clarification of the uniqueness in function. Such answers were "*There is a relation between students and their books*", "*The prices of vegetables depends on the production*", "*We correspond the marks of girls in a classroom to those of boys*". Examples presenting an equation instead of a function were coded as X4. "*There are 2x boys and 3y girls in a classroom and all the children are 60. If the boys are 15 we can calculate the number of girls*", "*Kostas has x number of toffees and John has double that number. How many toffees do the two friends have?*". The last category X5 included answers, which were ambiguous and in addition they did not define any variables or sets, and referred to general transformations of real world. Such answers were "*Health depends on smoking*", "*Success in a test depends on the hours of studying*", "*In the relation of children and parents, the children are the depended variable and parents the independent variable*".

(ii) Percentages

For the purposes of the present study we will only refer to the results, which show the strongest trends among the students. Question 1, requiring the recognition of verbal examples of function, was answered successfully by around 50% of the students, and this percentage was almost uniform for all the four parts of the same question. On the contrary, in Question 4 concerning the recognition of function given in an algebraic expression, the percentages varied between the different parts of the same question. The linear function $2x+y=0$ was recognised by 73% while 65% of the students answered that the equation of a circle $x^2+y^2=25$ presents a function. In Question 6, which concerned the recognition of a function when given in a Cartesian graph, the most difficult part was the line $y=4/3$ which was recognised as a function only by the 27% of the students, since it was treated in identical way with the line $x=-3/2$. In the same question, the discontinuous linear function of Q6e was recognised only by 31% of the students. It can be asserted that the majority of students appear to identify the stereotypical forms familiar to them from high school as functions.

(iii) Gras' s Implicative Analysis

From the similarity diagram shown in Figure 1, it ensues that there is a connection between four small groups Gr1, Gr2, Gr3, Gr4 that comprise the bigger cluster A. From these subgroups, the "strongest" is Gr2 formed around variables D1 and X2 that present the premier similarity (0,99999). That means that students who give an approximately correct definition (D1) in Question 9, give an example of a function one-to-one (X2) in Question 10. Around this strong subgroup the answers to questions Q6d and Q6e are linked. These are the questions asking the recognition of

some non-conventional cases of functions presented graphs (Q6d was a graph not representing a function and Q6e presented the graph of a discontinuous linear function). Finally this subgroup is completed with the answers in Question 2 (Q2), which concerns the translation from a verbal representation of a piecewise function to the algebraic form

Figure 1: Similarity diagram of the observed variables
Note: Similarities presented with bold lines are important at significant level 99%.

Around the strong group (Gr2) three other subgroups are organised (Gr1, Gr3 and Gr4), which concern the answers to the four parts of Question 8 (Q8a, Q8b, Q8c, Q8d) that is the recognition of functions presented in the form of mapping diagrams. The high similarity of this group (0,997) indicates that mapping diagrams are confronted in the same isolated way. The two groups Gr2 and Gr3 compose a new strong subgroup with degree of similarity 0,899. The subgroup Gr4 is further connected with the strong connection of subgroups Gr2 and Gr3, which include the answers to the other parts of Question 6 (recognition of function given in algebraic form). Conclusively the connection of subgroups Gr2-Gr3-Gr4 creates a group of answers, which show a conceptual approach to function. In other words the behaviour of the students to the definition and to the provision of an example of function has a predictive character in terms of their behaviour to functions when they are represented as graphs and diagrams.

Group Gr2-Gr3-Gr4 connects with Gr1 that includes the answers to Question 1, that is the recognition of functions when they are presented in verbal form. Finally this whole group (Gr1-Gr2-Gr3-Gr4) connects with the most "extraordinary" examples of Question 4 (Q4e and Q4f) that refer to recognition of functions in algebraic form. These are connected with the group that gave a correct example with the use of discrete elements of sets. This is the first "supplement" of group A.

The second supplement of strong connections is embodied by definition D2 and D3 and examples X3 and X4 that illustrate a vagueness or limited idea of the definition and the examples of function. These variables connect with answers to questions Q4c and Q4d, which are treated in that way that shows the wrong belief that in an algebraic form of a function symbols x and y must always appear. The third supplement is the group with the most doubtful idea about the notion of function since it includes D4 and X6 (i.e. those students that did not attempt any definition or example of function). Furthermore, this group is justified from high school direction, i.e. the students who have followed direction of classical studies. The third supplement behaves as an autonomous subgroup and consists of the answers that show absence of definition or example with a group of different questions that all have directly or indirectly a linear-algebraic character (Q4a, Q4b, Q3a, Q5a, Q7a). Also variable X1b (examples of function with the use of discrete elements of sets) is also connected with this group. The students that give answers that belong to the last group appear to have the misconception that function is just a linear relation.

Conclusively the strongest similarities in the diagram are (a) among responses providing correct definition and examples of functions and are mainly attributed to the students of mathematics department and (b) among responses giving no or very ambiguous answers and are attributed to the students of the education department, who come from the classical high school direction.

CONCLUSIONS

The study has revealed three strong trends in the ideas of students for function. The first is the identification of "function" by a large percentage of students with the more specific concept of "function one-to-one". The idea of uniqueness is particularly condensed and leads to identification of function as one-to-one function. Although this idea works for a wide range of situations and problems involving functions, it becomes a strong obstacle for the understanding of function as a wider concept. The second trend is the idea that "function" is an analytic relation between two variables (as it worked historically, initially with Bernoulli's definition, and more clearly with Euler's) and it is apparent in the way students define function and the examples they give. The third trend is that "function" is connected with a kind of diagram, either a Cartesian graph or a mapping diagram. On the contrary, when dealing with algebraic expressions the clear understanding of the definition of function is not essential; students respond to this latter form through certain stereotypical behaviours.

In respect with the two research questions, students' ideas are organised around two poles. The first is that of the conceptual understanding of function, which strongly connects with representations in the form of mapping diagrams and Cartesian graphs, and therefore has a higher level of success when dealing with most of the representations of functions. The other is the one dealing with function as a completely ambiguous relation, which connects with stereotypic forms of function that can be easily identified. Further research is essential in order to examine whether the formation of the above two poles may be modified through appropriate instruction.

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