PROPORTIONAL REASONING OF QUASI-ILLITERATE ADULTS

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The main purpose of this paper is to describe the answers given by adults without primary schooling to different ratio- and rate-comparison tasks. The framework and the analysed data are part of an ongoing research, in which the responses of subjects of different ages and schoolings are studied. The behaviour of quasi-illiterate adults could throw some light on the effect of school on proportional reasoning in normal conditions; evidence will be shown regarding the similarities of their phenomenological behaviour with the one of people with regular schooling, especially the influence of number structure and context upon proportional reasoning.

Part of an ongoing research on the strategies used by subjects of different ages and schoolings when faced to different kinds of ratio comparison tasks is reported in this paper. In the part here conveyed, we are concerned with the following question: Is proportional reasoning something developed exclusively at school, or does daily life provide the means to it? This question is difficult to answer in developed countries where all the population has a minimum of several years of schooling in which proportionality is taught. However, in Mexico about 20% of the adult population has none or very little schooling, which of course is nothing to be proud or glad about, but allows us to consider illiterate adults as epistemic subjects who could lead us to a tentative answer to our question.

PREVIOUS WORK: A FRAMEWORK

As stated above, we are investigating the strategies used by different subjects when faced to different kinds of ratio comparison tasks. In Alatorre and Figueras (2003) we explained what it is meant by "different kinds of ratio comparison tasks" and described the interview protocol used in the experimental part of the research. Some of the categories for classifying the questions, as well as the categories (strategies) for interpreting the answers, stem from a framework presented in Alatorre (2002). A succinct summary of both papers will be sketched here; the reader is referred to them for a more complete account.

Among the problems calling for proportional reasoning, those in which the task is a comparison of ratios or of rates can be classified according to three issues: context, quantity type, and numerical structure. Table 1, taken from Alatorre and Figueras (2003), proposes a joint classification according to the first two; it blends together the classifications proposed by several authors (Freudenthal, 1983; Tourniaire and Pulos, 1985; Lesh, Post and Behr, 1988; Schwartz, 1988; Lamon, 1993).

Rate problems: couples of expositions		Intensive quantity surging from two quantities: both discrete, both continuous, or one of each type
Part-part-whole problems: couples of compositions	Mixture	One quantity, discrete or continuous
	Probability	One quantity, discrete or continuous
Geometrical problems: couples of Σ-constructs		Two continuous quantities

Table 1: Taxonomy of ratio comparison tasks
according to context and quantity types

The third issue is the numerical structure. Before describing it, the notation used in this paper will be presented. In a ratio or rate comparison there are always four numbers and two "objects" (1 and 2) involved. In each object there is an antecedent "a" and a consequent "c", and thus the four numbers may be written in an array, which is an expression of the form $(a_1,c_1)(a_2,c_2)$. Also of interest may be the totals t=a+c, the differences d=a–c, and the part-whole quotients p=a/t. Alatorre's (2002) proposition is a classification of all arrays in 86 different situations according to 17 different "combinations" –successions of results when an order relationship is established in the array between the pairs of numbers t, a, c, d, and p–, and 17 different "locations" –non-ordered pairs of the following alternatives for both quotients of the array: *n:* nothing (p=0); *l:* lose (0<p<½); *d:* draw (p=½); *w:* win (½<p<1); and *u:* unit (p=1)–. The 86 situations can be grouped in six difficulty levels, labelled I to VI. Because of space limitations, for the purpose of this paper only three difficulty levels will be used: L1 (65 situations belonging to levels I, II, and III), L2 (3 situations belonging to level IV), and L3 (18 situations belonging to levels V and VI). The description of L1, L2, and L3 will follow the next paragraph.

In the previous paragraphs a description of the classification of ratio-comparison problems was given. Here follows a classification of the strategies used by subjects in their answers to such problems. Alatorre's (2002) framework, as presented in Alatorre and Figueras (2003), is to be used. Strategies can be simple or composed; in turn, simple strategies can be centrations or relations. Centrations can be on the totals CT, on the antecedents CA, or on the consequents CC. Relations, either "within" or "between", can be order relations RO (when an order relationship is established among a and c elements of each object and the results are compared), or subtractive relations RS (additive strategies), or proportionality relations RP. (For the purposes of this part of the research, RP relations were decomposed in several categories, which will be described further on). Composed strategies of two or more simple ones can be conjunctions *X&Y* (*X* and *Y* dominate), exclusions *X¬Y* (*X* dominates), compensations *X*Y* (*X* dominates), or counterweights *X⊥Y* (neither dominates). Some examples will be given further on. Strategies may be labelled as correct, sometimes depending on the situation (combination and location) in which they are used. Correct strategies are RP in all situations; RO in *wl*, *wd*, or *dl* locations; CA in locations with *n*; CC in locations with *u*; and, in some situations, some composed strategies that can be considered as theorems in action (see e.g. Vergnaud, 1981).

The three difficulty levels mentioned before refer to which correct strategies may be applied. Grouped in L1 are all the situations where, in addition to RP, all other correct strategies may be used. In L2 and L3 only RP can be used; the difference among them is that L2 consists of situations of proportionality (both ratios or rates are the same), and L3 consists of situations of non-proportionality.

METHODOLOGY

For the part of the research reported in this paper, a case study was conducted with six adults. They were students at a Centre for Adult Education in Mexico City; Reyna, Zoraida, and Ubaldo (aged 17, 49, 65) were learning to read and write, and Luisa, Dalia, and Toñita (aged 24, 25, 51) were studying a correlative primary school. All had less than the equivalent to four years of schooling. Four worked as housemaids, Toñita owned a small shop, and Ubaldo was a builder. They were interviewed for a time between 60 and 90 minutes, and the sessions were videotaped.

During the interviews, subjects were posed several questions in each of eight sorts of problems, which were Rate problems and both kinds of Part-part-whole problems (in this research Geometrical problems are not dealt with). Table 2 describes them.

CONTEXT			Objects	Antecedent	Consequent	Question
Rate	N	Notebook problem	Stores	Notebooks (d)	Coins (d)	In which store are the notebooks cheaper?
	B	Blocks problem	Walking girls	Blocks (x)	Time (x)	Which of the two girls walks faster?
	Y	Yards problem	School-yards	Children (d)	Yard squares (x)	Which of the yards is more cramped with children?
	L	Lemonade problem	Jars	Lemons (d)	Cups with sugared water (x)	In which jar is the lemonade's taste stronger?
Part-part-whole	Mixture E	Exams problem	Exams	Correct answers (d)	Incorrect answers (d)	In which exam did the student do better?
	Mixture J	Juice problem	Jars	Concentrate glasses (x)	Water glasses (x)	In which jar does the mixture taste stronger?
	Probability M	Marbles problem	Bottles	Blue marbles (d)	Yellow marbles (d)	In which bottle is it more likely to get a blue marble?
	Probability S	Spinner problem	Spinners	Blue sectors (x)	Yellow sectors (x)	In which spinner is a blue sector more likely to be marked?

Table 2. The eight problems of the protocol (d=discrete; x=continuous)

Each of the problems was posed in different questions according to numerical structure. Fifteen such questions were designed, five in each of the difficulty levels; they are displayed in Table 3. All the problems may be posed in each of 15 questions, except for question 7, which has no sense in problems B or Y (see table 2). Questions 1 to 10 were posed to all subjects; questions 11 to 15 only to those who used RP in

Level	Question numbers and their arrays $(a_1, c_1)(a_2, c_2)$				
L1	1 (2, 3)(2, 3),	2 (1, 4)(3, 2),	3 (2, 3)(2, 3),	6 (2, 2)(3, 2),	7 (3, 3)(2, 0)
L2	5 (3, 3)(1, 1),	8 (2, 1)(4, 2),	10 (3, 6)(1, 2),	12 (4, 6)(2, 3),	15 (8, 4)(4, 2)
L3	4 (2, 1)(3, 2),	9 (2, 5)(1, 3),	11 (5, 2)(7, 3),	13 (3, 2)(5, 3),	14 (2, 4)(3, 5)

Table 3. Numerical structure of the fifteen questions

previous questions of the same problem. The problems were presented to subjects in a graphical form, which can be consulted in Alatorre and Figueras (2003); Figure 1 is an example. The problems were posed in this order: M, N, J, Y, S, L, E, B (see table 2). Within each one, 10 to 15 questions were posed. In each of them, the subjects were asked to make a decision (left side, right side, or "it is the same") and to justify it.
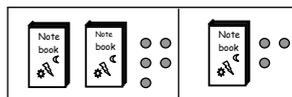

Figure 1. Question N-9

RESULTS AND ANALYSIS

A total of 513 answers was obtained; 424 (83%) of them were classified using the strategies system described above, and the rest either consisted of a decision without a justification (35), or had a justification that was only a description (27), or consisted of solution mechanisms different from the strategies described before (27). How these non-classifiable answers were handled will be described below.

Two phases of analysis were undertaken: first a quantitative, then a qualitative one. In order to make a quantitative analysis possible, one point was given to all correct strategies, and ½ point was given to answers that could be incomplete expressions of correct theorems in actions. Also, ½ point was given to all non-classifiable answers that fulfilled the following conditions: correct decision and either no mechanism or a mechanism that could eventually become correct (such as arithmetic or geometric approximations). Then, for each subject a score was obtained for each group of context and difficulty level, and expressed as a percentage of the questions answered by the subject in that group. The results are shown in Figure 2.
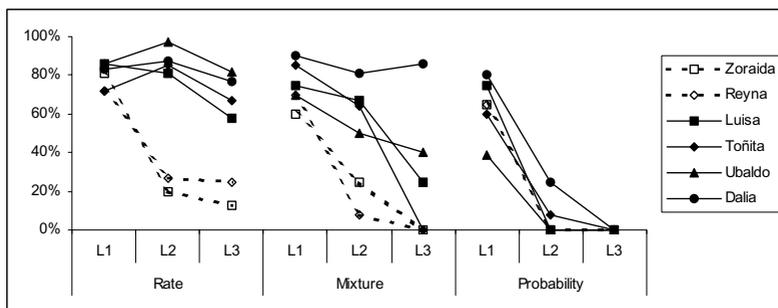

Figure 2. Subjects' performance

As Figure 2 suggests, the six subjects can be divided in two groups. Both groups obtained their best results in Rate and their worse in Probability problems, but with different behaviour. Subjects of the first group (Zoraida and Reyna, dotted lines) only answered fairly well the L1 questions of all contexts and dropped their performances in levels L2 and L3. The latter proved to be very difficult for them; their performance averaged 19% in Rate problems and was null elsewhere.

The remaining subjects had also their lowest results in L3 of all contexts, but only in Probability problems was it null for all of them, and elsewhere the decrease was not as marked as in the first group. Compared to the first group, subjects of the second one had a tendency to obtain better results in all L1, and significantly higher performances in L2. All of the subjects except Luisa even had better results in L2 than in L1 in the Rate problems, showing that they could recognise and appropriately solve situations of proportionality.

For the qualitative phase of analysis, the strategies used by subjects were studied. All sorts of strategies were used; here are some examples of centrations, RO and RS:

Reyna:	I choose the right side, because there're more children and the space is smaller (question Y-2, correct composed strategy CA & CC).
Ubaldo:	She did better on the left side because of the correct ones, although she has six wrong answers (question E-10, incorrect composed strategy CA ¬ CC).
Reyna:	I choose the right side, because it has more concentrate than water, and in the other one they're the same (question J-6, correct simple strategy RO).
Luisa:	It's the same, because in both there are more blue marbles and fewer yellow ones (question M-9, incorrect simple strategy RO).
Zoraida:	The right side, because there're two more lemons than cups, and on the left side there's only one more (question L-8, incorrect simple strategy RS).

As for the correct proportionality relations, it became necessary to focus on the different kinds of RP. Four kinds were identified, which are described below:

RPM ("Multiples"): The subject realizes that there are multiples among the numbers of the array, either within an object or between objects. Example:

Dalia:	It's the same: The left side is half as much as the right side (question E-8).

RPG ("Groupings"): The subject uses groups of specific amounts of antecedents and consequents, identifies them as appearing once or several times in each object, and compares the remaining elements (if any). Example:

Toñita:	I choose the right side, because on the left side there's one and a half lemon for a cup, on the right side one and a half for one, one and a half for another one, and one and a half for another one. And there's half a lemon more (question L-13).

RPE ("Equalizing"): The subject executes a physical or a mental action of multiplying or dividing one of the objects by a certain quantity and by doing so either

equalizes both objects, or he/she equalizes the antecedents (or consequents), which permits the comparison of the consequents (or antecedents). Example:

Ubaldo: I choose the left side, because in one more minute the girl would walk four blocks, and the one on the right [only] walks three blocks in those two minutes (question B-4).

RPR ("Rate or ratio comparison"): The subject calculates in each object the rate or the ratio a:c, the part-part quotient a/c or the part-whole quotient a/t. This is generally accomplished through the calculation of the unity value. Then he/she compares both results. Example:

Luisa: I choose the left side, because on the right a notebook costs three coins, and on the left it would cost less than three coins, about two and a half coins (question N-9, see Figure 1).

The qualitative analysis is based on two considerations: whether the correct strategies were RP (and, in that case, what kind of RP), and the classification of incorrect answers. As with the quantitative analysis, the same two groups can be identified.

In the first group, Zoraida and Reyna's behaviour was characterized by the use both of centrations, either in simple or in composed strategies, and order relations. These are strategies that may lead to correct answers in L1, but necessarily lead to incorrect ones in L2 and L3. In the case of these two subjects, these strategies account for all of the correct answers in L1 and, mainly centrations, for most of the incorrect ones in L2 and L3. The few exceptions are some additive strategies and, in the Spinner problem, a mechanism of choosing the spinner with a bigger chunk of blue colour, such as the left side in Figure 3. In levels L2 and L3 each of the subjects had six scarce correct RP strategies, in the Notebooks and Lemonade problems, and once each in the Blocks and the Juice problems. These few attempts at proportional reasoning strategies were mainly of the RPM kind in L2 proportionality situations. Reyna displayed as well a couple of RPEs and Zoraida a couple of RPRs.
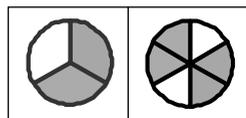

Figure 3. Question S-8

In the second group, the rest of the subjects showed a much richer behaviour, characterized by what seems to be a quest for the easiest correct strategy. In L1, the four subjects profusely used the simple and correct RO relations and theorems in action; however, all of them used at least once a RPR strategy. All four strategies leading to the answer "it is the same" in L2 were used: mostly RPR, but also RPG, RPE, and RPM (in this order); and all but RPM (in the same order) were used to choose one of the objects in L3. Save for Ubaldo, who never used RPM, all four subjects used all four RP strategies, in the same frequency order. This was especially notorious in the Rate problems, where RPR accounts for 64% and RPG for 27% of all RPs. As for the distribution among contexts, all four subjects used RP strategies in the four Rate problems, with small variations (RP were more frequently used in the Lemonade problem and less frequently used in the Yard problem), but used them

much more sparingly in Part-part-whole problems: Two of the subjects (Ubaldo and Luisa) used them only in the Juice problem, one (Toñita) used them as well twice in the Spinner problem, and only one (Dalia) used them in the Exams problem. The use by these subjects of RP strategies in Probability problems was very scarce (Spinners) or null (Marbles).

An analysis of the incorrect answers of the four subjects of the second group leads to the following. In the Rate problems, only one of the subjects (Luisa) used centrations and RO relations incorrectly in L2 and L3; all other incorrect answers were due to either additive relations RS or incorrect attempts at some of the RP strategies, mainly with arithmetic mistakes. This was also the case in the Juice problem. In the Exams problem all four subjects incorrectly used centrations, and some of them used incorrect RO and RS relations. The abundant incorrect answers in both Probability problems range from centrations to RO relations to the mechanism of big chunks described above but also the inverse mechanism: choosing the side where the blue sectors are more scattered, such as the right side in Figure 3. There were also some mechanisms due to misconceptions of randomness, such as the following:

Ubaldo: It's the same. If it's my luck, I win. I win 'cause I win. If not, I don't win; even with twenty blue marbles and one yellow one, I don't (question M-4).

CONCLUSIONS

Both phases of the analysis lead to similar interpretations. Some of the subjects with little or no schooling approach the rate- and ratio-comparison problems in ways similar to that reported in the literature about young children (e.g. Noelting, 1980): their first choice being centrations, they only succeed where centrations lead to the correct answer, and fail elsewhere. This is to say, their performance depends deeply on the numerical structure of the questions, and they succeed where it allows non-proportional reasoning. However, they do occasionally produce one form of proportional reasoning, mostly in proportionality situations.

Another group of subjects does produce an assortment of forms of proportional reasoning. They succeed in the most easy questions mainly by using correct strategies different from the proportionality relations, but they also may succeed in questions where the only way to reach a correct answer is with the use of proportional reasoning. Here they used different kinds of strategies, apparently in a search for the easiest one. The performance of these subjects depends on the context: it is fairly good in all Rate problems, but decreases in Part-part-whole problems. The relative success in Mixture problems is due mainly to success with the Juice problem, since the other Mixture problem, Exams, had a very low success rate, probably due to the fact that exams are alien to the experience of these subjects, who are only beginning their schooling. This corroborates that familiarity with the problem is one important factor for success in proportionality problems (Tourniaire and Pulos, 1985; Lamon, 1993). Both Probability problems had also very low success rates, which may be due to the combined effect of lack of familiarity and the difficulties of randomness.

This study confirms some of the findings reported in the literature of proportional reasoning (Tourniaire and Pulos, 1985): that there are important effects of numerical structure and of context, that Rate problems are easier than Part-part-whole ones, and that familiarity with the problem is crucial. Therefore, these results seem to be independent of the schooling of the subjects considered.

The responses of the quasi-illiterate subjects who participated in this study resemble those of some university students (Alatorre, 2000): Some use non-proportional strategies and get entangled in number structures which are more complex than the easiest ones, whereas some are able to use proportional strategies and surmount the numerical difficulties. The latter obtain much better results than the former, although their performance is context-dependent.

The literature on the subject has long ago demonstrated that schooling is not a sufficient condition to reach an appropriate proportional reasoning. The fact that daily life has provided some of the quasi-illiterate subjects with a fairly good performance, at least in Rate and some Part-part-whole contexts, seems to suggest that schooling might also not be a necessary condition for proportional reasoning.

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