CHARACTERIZATION OF STUDENTS' REASONING AND PROOF ABILITIES IN 3-DIMENSIONAL GEOMETRY

Angel Gutierrez [1] John Pegg [2] Christine Lawrie [2]

[1] Dpto. de Didactica de la Matematica. Universidad de Valencia. Valencia (Spain)

[2] Centre for Cognition Research in Learning and Teaching. University of New England. Armidale, NSW (Australia)

In this paper we report on a research aimed to identify and characterize secondary school students' reasoning and proof abilities when working with 3-dimensional geometric solids. We analyze students' answers to two problems asking them to prove certain properties of prisms. As results of this analysis, we get, on the one side, a characterization of students' answers in terms of Van Hiele levels of reasoning and, on the other side, a classification of these answers in different types of proofs. Results from this research give directions to grade and organize secondary school instruction on 3-dimensional geometry.

INTRODUCTION

The little time devoted by primary and secondary school teacher to teach space geometry is parallel to the little (and quite often wrong) knowledge students show when they have to solve problems in this geometry field. There is an active international research agenda interested in solving several questions related to this problem, like relationships between visualization abilities and the learning of space geometry (Gutierrez, 1996; Kwon et al., 2001; Malara, 1998; Meissner, Pinkernell, 2000), subjects' reasoning processes (Gray, 1999; Guillen, 1996; Lawrie et al., 2002; Meissner, 2001; Owens, 1999), students' knowledge and ways of learning (Jirotkova, Littler, 2002; Lampen, Murray, 2001; Lawrie et al., 2002), improvement of teaching strategies (Lavy, Bershadsky, 2002), benefits for students of using manipulatives (Jirotkova, Littler, 2002) or software (Kwon et al., 2001), problem solving (Lampen, Murray, 2001; Owens, 1996; Stylianou et al., 1999), or theories framing research and curriculum development (Gutierrez, 1996; Owens, 1999; Saads, Davis, 1997).

The research reported in this paper is part of this agenda. Its central focus is to analyze students' level of reasoning and their ability to conjecture and prove in the context of space geometry. As a first step, we have designed a test aimed to provide information about the kinds of outcomes produced by secondary school students when solving problems of proof or conjecture and proof. The result obtained after the administration of the test is a set of students' responses reflecting different Van Hiele levels of reasoning and several types of empirical and deductive proofs.

THE FRAMEWORK AND RELATED LITERATURE

Three elements integrate the theoretical framework of this research: i) The concept network of prism is the mathematical content base of the problems solved by students. ii) The Van Hiele levels of reasoning, as characterized in Gutierrez, Jaime (1998) and applying the paradigm of evaluation defined in Gutierrez, Jaime, Fortuny (1991), are used to identify students' reasoning while solving the problems. iii) The categories of mathematical proofs defined in Marrades, Gutierrez (2000) are used to classify the types of proofs produced by students.

i) The two problems we posed to students deal with prisms and some of their elements and properties (see below the statements of the problems). The main concepts and properties necessary to solve the problems are:

* A <u>prism</u> is a polyhedron having two parallel congruent faces (bases) linked by parallelogram faces (side-faces). Prisms can be <u>right prisms</u>, and <u>oblique prisms</u>.
* In any prism, all the side-edges are parallel and congruent. In a right prism, all the side-edges are perpendicular to the bases (and to all the base-edges), and all the side-faces are rectangles. In an oblique prism, no side-edge is perpendicular to the bases, and at least a side-face is not a rectangle.
* A <u>diagonal</u> is a segment joining two non-consecutive vertices of a polyhedron. Diagonals can be <u>face diagonal</u> and <u>space diagonal</u>.
* A n-gonal prism has n+2 faces, 2n vertices, 3n edges, and 2n(n-2) diagonals, n(n-1) of them being face diagonals, and n(n-3) of them being space diagonals.

ii) There is extensive literature describing the characteristics of Van Hiele levels. Some refer to space geometry (Gray, 1999; Guillen, 1996; Gutierrez, 1992; Lawrie et al., 2000, 2002; Owens, 1999; Saads, Davis, 1997). The main characteristics of Van Hiele levels referred to the context of prisms and diagonals are stated below. We used these descriptors to analyze students' answers and identify their levels of reasoning.

<u>Level 1</u>: Students are able to draw some diagonals in a given prism, but they are not exhaustive nor can induce a general relationship. Their explanations or justifications are just a description of what they have drawn.

<u>Level 2</u>: Students induce a formula for the number of diagonals of a n-gonal prism after drawing and counting the diagonals in a few prisms, and they justify it just by summarizing the data they have considered. Students can use the formula they have obtained to calculate the number of diagonals or sides of a given prism.

Students prove that a given conjecture is true (false) by drawing a figure as an example (counter-example) to show that the conjecture is (is not) verified. Their justifications are just a description of what they have drawn. In particular, students use a square (cube) as a counter-example for a rectangle (right prism).

Some times students prove that a given conjecture is true by providing a deductive argument that really proves the converse of the given conjecture.

<u>Level 3</u>: Students induce a formula for the number of diagonals of a n-gonal prism in the same way as those reasoning in level 2, but in this level proofs are abstract

deductive informal arguments (some times based on a specific example drawn) connecting data with the conjecture.

Level 4: Students induce a formula for the number of diagonals of a n-gonal prism by first drawing some specific examples, and then writing a proof. In this level proofs are abstract deductive formal arguments connecting data with the conjecture.

iii) Balacheff (1988) and Harel, Sowder (1998) proposed two well known categorizations of mathematical proofs. More recently, Marrades, Gutierrez (2000) proposed a new set of categories which integrates and expands those defined by the above mentioned authors. We have used the latest categorization to analyze and classify the proof produced by the students participating in our research. Schematically, the categories defined in Marrades, Gutierrez (2000) are:

In empirical proofs examples are the argument of conviction. There are three classes, depending on the *way students select* the examples: Naive empiricism (the conjecture is proved by showing that it is true in one or more, randomly selected, examples), crucial experiment (the conjecture is proved by showing that it is true in a carefully selected, example), and generic example (the proof is based on a specific example, seen as a representative of its class, and it includes explicit abstract justifications.

Each class of empirical proofs has several types corresponding to *ways students use* the selected examples in their proofs: Perceptual proofs are naive proofs involving only visual or tactile perception of examples. Inductive proofs are naive proofs including mathematical elements or relationships. Example-based proofs consist only in showing the existence of an example. Constructive proofs consist in describing the way of getting the example. Analytical proofs consist in using properties empirically observed in the example. Intellectual proofs are based on empirical observation of the example, but the they mainly use abstract properties of the example.

Deductive proofs consist on the use of abstract deductive arguments. There are two classes of deductive proofs, depending on whether students use an example or not: In a thought experiment a specific example is used to help organize the proof. A formal proof is based on mental operations built without the help of examples.

Each class of deductive proofs has two types depending on the *styles of proof* made: Transformative proofs are based on mental operations producing a transformation of the initial problem into another one. Structural proofs consist in sequences of logical deductions derived from the data and axioms, definitions or accepted theorems.

THE EXPERIMENT

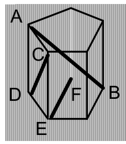
To get information on secondary school students' levels of reasoning and proof abilities, we designed an experiment based on the development and administration of a paper and pencil test to evaluate students' behavior and content knowledge in several areas of space geometry. The test has seven items, six of them having the structure of super-item (Collis et al., 1986). The contents of the seven items are: Identification, description, and characterization of solids and their parts (faces, edges, vertices, diagonals); Classification of solids; Cross-sections of solids; Nets of solids; Conjecturing and proving properties of solids.

In this paper we analyze the answers of 299 students from several mixed ability class groups in grades 7 to 11 (aged 12 to 17 years) from three secondary schools in a rural city of New South Wales (Australia). The test was administered to the whole class groups. The Table below summarizes the number of students in each grade and school.

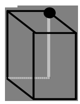
	7th grade	*8th grade*	*9th grade*	*10th grade*	*11th grade*
High School N	12	26	18	35	27
High School O	49	12	23	--	33
High School P	28	--	16	--	20
Total	89	38	57	35	80

We have presented elsewhere the results of the items dealing with nets and cross-sections of solids (Lawrie et al., 2000, 2002). In this paper we analyze the answers to two items asking students to obtain and prove conjectures about prisms:

Item A: a) Remember that a diagonal of a polyhedron is any segment joining two non - neighbouring vertices of the polyhedron. In the figure you can see a polyhedron (a pentagonal prism). Segments AB, CD, and EF are some of its diagonals. Draw three more diagonals of this polyhedron.

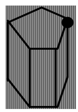


b) How many diagonals has a n-gonal prism (that is, a prism whose base is a n-sided polygon)? Explain, justify or prove your answer.



c) How many diagonals starting from the marked vertex has a rectangular prism? Explain your answer.

d) How many diagonals starting from the marked vertex has a pentagonal prism? Explain your answer.



e) How many diagonals has a n-gonal prism (that is, a prism whose base is a n-sided polygon)? Explain, justify or prove your answer.

f) What prism has exactly 48 diagonals? Explain, justify or prove your answer.

Item B: Tell if the following statement is true or false, and give an explanation, justification, or proof for your answer:

"If all the side-edges of a prism are perpendicular to the base, then all its side-faces are rectangles".

The statement is .... Explain, justify, or prove your answer.

Question A-a is a reminder of the definition of diagonal of a polyhedron. Question A-b states the main problem without any help. To solve it, students need to reason at levels 2, 3 or 4, producing different answers depending of their level of reasoning. For those students not able to solve the problem in A-b, questions A-c and A-d are a prompt showing the main clue to elaborate the conjecture and its proof. To answer these questions, only reasoning of levels 1 or 2 is required. Afterwards, question A-e states again the main problem, to check if students are able to solve it after working on the clues. Now, students need to reason at levels 2 or 3. As the answer to question A-e has been guided by questions A-c and A-d, reasoning of level 4 is not required to answer it. Finally, question A-f asks students to apply the result they have obtained in A-b or A-e. Only reasoning of level 2 is required to answer this question.

Item A was presented to students split into two pages, the first one containing A-a and A-b, and the second one containing A-c to A-f. In this way, students do not see the clues while they are trying to solve the problem for the first time (A-b).

Item B asks to prove a given property of right prisms. This is a harder problem, since no prompt is provided to students. Possible answers range from just a drawing followed by a comment to a formal proof, so reasoning of levels 2, 3 or 4 is required.

ANALYSIS OF RESULTS

The objective of the research is to identify kinds of answers produced by secondary school students, so we do not include here quantitative information about frequencies of answers. Below we present examples of the main kinds of answers produced by the secondary school students in the sample.

Answers to question A-b	**V.H. level**	**Type of proof**
An n-gonal prism has 2n diagonals. E.g. a triangular prism has 6 diagonals (2x3 sides). A rectangular prism has 8 diagonals (2x4 sides). [included pictures of a triangular and a rectangular prism and the diagonals of their side-faces]	2	Inductive Naive empiricism
[after counting, with some mistakes, the number of diagonals in several polygons: 4 sides-2 diagonals, 5-5, 6-8, 7-14, 8-19] *From 4-2 and 5-5 obtains d = 3n-10. An n-sided polygon has 3n-10 diagonals. An n-sided prism has two bases (i.e. 2(3n-10)) and 2n diagonals in the rectangles. Then d = 8n - 20 for any prism.*	3	Intellectual Generic example

The first answer exhibits a level 2 reasoning, since a general formula has been induced from some examples. The proof are the examples used to induce the formula, so it is a naive empiricism proof. Furthermore, the student has used the examples drawn to get mathematical information, so the proof is of the inductive type.

The second answer begins with typical level 2 reasoning (inducing the number of diagonals of a polygon) but then it shifts to level 3 reasoning, a generic abstract deductive process to obtain the formula for a n-gonal prism. This proof is built on properties observed in the examples and then stated abstractly, since they refer to a n-gonal prism, so it is an intellectual generic example proof.

Answers to questions A-c, d, e	**V.H. level**	**Type of proof**
The student drew the diagonals in the given prisms and wrote the numbers of diagonals. No answer to question A-e.	1	- - - -
[In A-c (A-d), the student drew 4 (6) diagonals] *There are 8 (10) vertices; 4 are excluded (the dot and the three others directly beside).* A-e: $d = n - 4$ [without any comment or proof]	2	Perceptual Naive empiricism
[In A-c (A-d), the student drew 4 (6) diagonals and wrote the numbers] A-e: *Each rectangular side has 2 diagonals, so 2n. There are 2 n-gonal bases, so 2n. The space diagonals are n(n-1). Then, d = n(n+3). This doesn't work for prisms where n²4.*	3	Analytical Generic example

Many students produced answers like the first one. They are able to draw and count the diagonals from a vertex, but they are not able to induce a general formula, so they are reasoning at level 1. In this case there is not a mathematical proof.

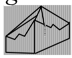
Students reasoning at level 2 usually solve correctly questions A-c and A-d, as they understand which vertices can/cannot be linked by a diagonal, like second answer. Then they try to induce a general formula to answer question A-e, although they do not provide a reasonable proof of such formula. This behavior is typical of level 2 reasoning, proofs being usually of types naive empiricism or crucial experiment.

The third answer is clearly different from the second one because in question A-e the student wrote a proof of the formula. It is an abstract generic description of the process of getting the formula, based on generalizing the two specific examples of A-c and A-d to a n-gonal prism, so it is an analytical generic example proof.

Students answered question A-f in a meaningful way only when they had obtained a general formula for the number of diagonals of a prism in previous questions. In such case, students used their formula to calculate the number of sides of the given prism, therefore exhibiting a level 2 style of reasoning.

Only a few students in grades 7 to 9 produced meaningful answers to item B, but there were more answers from students in grade 10 and especially grade 11.

Answers to item B	**V.H. level**	**Type of proof**
False: They [the side-faces] *could be squares.*	2	Counter-ex
True: [the student drew a rectangular right prism] *Side edges are perpendicular to the base. Side faces = rectangles.*	2	Ex.-based Crucial experiment
True: The solid could be a rectangular or square prism or a cube. Since squares can be rectangles and the angles in squares and rectangles are 90¡, all side-edges are perpendicular to the base.	2	Converse

False: When you look at more complex solids this statement becomes untrue, like with a dome as structure  *. The side-edges start being perpendicular but then change. This doesn't really agree with the statement. Or a solid as*  *where they are perpendicular but the top of the rectangle is cut off to give jaggard edges. This also doesn't agree with the statement.*	3	Counter-example
True: Prove: That side edges perpendicular make rectangular side faces. Data: Above [the statement of the item]. *Proof: The side edges make right angles with the base edges. The side edges are parallel (all at 90¡ to the base).* ∴ *The side faces are parallelograms with 90¡ angles.* ∴ *The side faces are rectangles.*	4	Structural Formal proof

Many students, like the first case, considered that squares are not rectangles, so they provided a cube or a right prism with square side-faces as a counter-example for the conjecture. Other students produced more elaborated answers, like the fourth case, drawing prism-like solids and analyzing them to show that the conjecture was false. These proofs cannot be classified into the categories defined in Marrades, Gutierrez (2000) because these categories refer only to proofs of the truth of a conjecture.

Some students made the usual mistake of proving the converse implication, like the one in the third case, showing that they still have not acquired the level 3 reasoning.

Finally, very few students produced formal proofs, like the last case, exhibiting level 4 reasoning. This proof is an example of structural formal proofs, since it does not include any drawing as auxiliary guide to build the deductive argument.

CONCLUSION

An overview of the answers obtained shows a quite complete range of answers in Van Hiele levels 1 to 3. On the contrary, we have only obtained a few answers in level 4, as could be expected from a sample of secondary school students. Research based on university students should be carried out to complete the catalog of answers for the higher levels of reasoning and deductive classes of proofs.

After completing the catalog of answers, the next step in this research program is to design and experiment teaching units focusing on the learning of geometric solids and the improvement of students' reasoning levels and proving abilities.

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