# GENERALISING ARITHMETIC: SUPPORTING THE PROCESS IN THE EARLY YEARS

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*The teaching and learning in algebra has been much debated. Traditionally early algebra has relied heavily on arithmetic. Recently our focus has changed to teaching algebraic thinking with arithmetic thinking. This paper explores the models that assist young students generalise the patterns of arithmetic compensation. A teaching experiment was conducted with two classes of students with an average of eight years and six months. From the results it seems that the use of unmeasured quantity models in conjunction with number models does assist students focus on the underlying generalizations inherent in the models presented.*

## INTRODUCTION

Traditionally early algebra learning has occurred as an extension of arithmetic. Current research continues to indicate that many students experience difficulties in moving from an arithmetic world to an algebraic world, and it seems that many of the difficulties students experience originate from a lack of an appropriate foundation in arithmetic (Carpenter & Franke, 2001; Warren & Cooper, 2001; Warren, 2002; Warren, 2003). The assumption has been that as part of everyday classroom arithmetic experiences using the four operations, students will induce the fundamental structure of arithmetic. But research suggests that they are not. One way of addressing many of these issues, and helping students to *do* algebra, is to involve students in patterns of generalised thinking throughout their education. The focus is away from computation and onto the underlying mathematical structure exemplified by the carefully chosen examples, with an aim of explicitly abstracting arithmetic structure. Malara and Navarra (2003) suggest that a way of distinguishing this difference in the early years is that algebraic thinking is about process whereas arithmetic thinking is about product (reaching the answer). This paper reports on the former, young students generalising arithmetic processes and patterns, and in particular arithmetic compensation.

We are interpreting arithmetic compensation as the idea that if A+B = C, then A-k+B+k=C, or A+k+B-k=C, (e.g., 13+34=47 then 13**-3**+34**+3**=47). In other words, if we increase/decrease one number by a certain amount we must decrease/increase the other number by the *same* amount for the answer to stay the same. Put simply, for Part + Part = Whole, if we keep the Whole constant and change the value of one of the parts, we must compensate for this by changing the other part by the same amount. An extension of this is that for problems such as A+B+C+D=E, if we increase/decrease one of the numbers then we must decrease/increase some or all the other numbers by an amount that is the *same* for the answer to stay unaltered.

Past research has shown that even in the early years students' prior experiences in number interfere with their ability to generalise patterns and the structure of arithmetic.

This is particularly prevalent in research conducted on their interpretation of the equal sign (Carpenter and Levi, 2001; Warren, 2001) and seems to be related to specific classroom activities and experiences.

In this research, we attempt to use measurement models in conjunction with number patterns to assist students move from arithmetic product to process. The underpinning theory is that number is an abstract concept and represents a quantity that may or may not be obvious. Commonly students begin their learning of number by counting discrete objects (Dougherty and Zillox, 2004). As they move through different number systems, algorithms and routines are commonly changed to deal with the new number set which results, it is conjectured, in students failing to develop a consistent conceptual base that can deal with all numbers as a connected whole.

To bridge this gap, we are following Davydov (1975) in beginning students' experiences on the basic conceptual ideas of mathematics through using unmeasured quantities before moving into number exploration. Daydov claimed that students should begin their mathematics program without number and explore physical attributes such as length, area, and volume, that is, attributes that can be compared. He hypothesised that this allowed young students to more effectively focus on the underlying concepts of mathematics, such as equivalence and non-equivalence without the interference of numbers. We are investigating how such explorations in conjunction with traditional early number experiences can enhance young students understanding of arithmetic processes.

## BACKGROUND

The data reported here is part of an Australian Research Council Linkage Grant, a three-year longitudinal multi-tiered teaching experiment. In this study, we are designing and trialling activities and teaching experiences with 8 year olds in five elementary schools. The aims of this large study are to investigate Years 3 to 5 student's abilities to reason algebraically, in particular, to represent, relate and change arithmetical situations in a general manner, to identify key transitions in their development of algebraic reasoning, to construct a model of young student's cognitive development with respect to algebraic reasoning, and to develop instructional strategies effective in facilitating young students' construction of algebraic reasoning.

In this paper, we describe a lesson given to two Year 3 classrooms. The lesson was designed to generalise the addition compensation rule. The specific aims of the lesson were to: (i) investigate models and instruction that begin to assist young students to generalize and formalize their mathematical thinking; and (ii) delineate thinking that supports the development of algebra understanding. The aim of this paper is to document and explain students' generalizations.

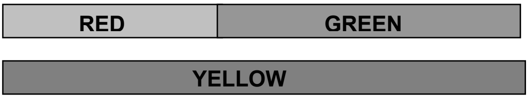
## METHOD

The lessons were conducted in two Year 3 classrooms from two middle class state elementary school from an inner city suburb of a major city. The sample, therefore, comprised 45 students, two classroom teachers and 2 researchers. These students had completed classroom experiences involving adding and subtracting two-digit numbers. We worked collaboratively with the teachers to trial teaching ideas and document student learning in relation to teaching actions. The trials of the lesson reported in this paper were those conducted by one of the researchers (teacher/researcher).

The lesson consisted of three phases, integrated so that there was a flow from one phase to the next, and was of approximately one hour duration. During the lesson three types of questions were continuously asked, namely, predicting, justifying, and generalising. The three phases are described below – a more detailed description of the lesson is in Warren & Cooper (2003).

**Phase 1 – Introducing the addition compensation rule using the length model**

The lesson began with a teacher/researcher led discussion. This initial discussion relied on developing an understanding of addition compensation using unmeasured quantities. In this instance the length model was chosen to represent these ideas.




First students were encouraged to invent names for the three strips of paper and express the relationship between the three strips, for example, (Length) Copt + (Length) Nopt = (Length) Lopt. Then a piece was cut off the **red strip (copt)** and the class was asked: *What would I have to do to the green strip (nopt) so that the length of the green strip plus the length of the red strip (copt) is still the same length as the yellow strip (lopt)? How much would I have to add to the green strip?*

This process was repeated a number of times using new strips each time with a number of students participating in the physical transformations of the length models. In all cases, the first length was reduced. The following discussion ensued,

Teacher: Can someone tell me what the generalization is.
Child 1: Cutting a bit of length off and replacing it with another bit of length.
Teacher: What is special about the other piece of length? What is the word we use?
Child 2: Equal
Teacher: What is another word for equal?
Child 3: Plus
Teacher: Yes we added something but what is another word for equal? What can you tell me about the two lengths [the one cut off copt and the one added to lopt]
Child 4: Same.

Students were then asked to write their own 'pattern rule' using their own language.

**Phase 2 Transferring from the length model to number.**

In this instance the set model acted as a mediator between unmeasured quantities and a symbolic world. For example, for 6+8, we explored what happened to 8 if we decreased 6 by 2, decreased 6 by 4 and so on. The teacher/researcher modelled the process using the set model, moving two counters from one part to the other part and discussing how the parts changed by equal amounts (e.g., 2) but the whole remained the same. Students were encouraged to model this process using counters. As they worked through these scenarios the students recorded their responses on a worksheet and checked their answers on the calculator. Figure 1 represents the worksheet used

| 6 + 8 = 14 | | | | |
|---|---|---|---|---|
| 6 | + | 8 | = | 14 |
| 4 | + | | = | 14 |
| | + | | = | 14 |
| | + | | = | 14 |
| | + | | = | 14 |

| ↓ | ↑ |
|---|---|
| 0 | 0 |
| -2 | |
| | |
| | |
| | |

*Figure 1* Worksheet for recording answers for the set model activity.

In previous research we have found that students tend to look down the table when searching for patterns, thus finding a vertical pattern (Warren, 1996, 1997). We conjectured that one way of overcoming this was to ensure that the numbers placed in the first column do not conform to a vertical pattern, for example, 6, 3, 4, 5, 2, 1 rather than 6, 5, 4, 3, 2, 1.

**Phase 3 – Moving into a number world**

The same process was used for exploring addition number sentences with larger numbers (e.g., 12 + 15 = 27 subtract 3 from 12, and 26 + 33 = 59 subtract 11 from 26) and extended to situations where the first number was increased. Some children experienced difficulties in making the transition to larger numbers and used counters to support their explorations. Others simply used the 'pattern rule' to reach answers (e.g., add 3 to 12 and take 3 from 15). At the completion of phase 2 and phase 3 students were once again asked to record their own 'pattern rules' in their language. The students were finally brought together as a group and asked to explain how the length activity was related to the number activity.

**Data gathering techniques and procedures**

The lesson was taught to each class by one of the researchers. The two trials occurred sequentially. During the teaching phases, the other researcher and classroom teacher acted as participant observers. In each instance the other researcher and classroom teacher recorded field notes of significant events including student-teacher/researcher interactions. Both lessons were videotaped using two video cameras, one on the teacher and one on the students, particularly focussing on the students that actively participated in the discussion.

The basis of rigour in participant observation is "the careful and conscious linking of the social process of engagement in the field with the technical aspects of data collection and decisions which that linking involves" (Ball, 1997, p311). Thus both observers acknowledged the interplay between them as classroom participants and their role in the research process.

At the completion of the teaching phase, the researcher and teacher reflected on their field notes, endeavouring to minimise the distortions inherent in this form of data collection, and come to some common perspective of the instruction that occurred and the thinking exhibited by the students participating in the classroom discussions. The video-tapes were transcribed and worksheets collected.

**RESULTS**

The two classrooms were different with respect to the students' abilities to complete the lesson, although there were many similarities in students' responses to teaching strategies. The second classroom had more difficulties than the first, particularly with Phase 3 and numbers increasing and decreasing, as the following excerpt shows. IT seems that they did not understand increasing and decreasing in relation to addition.

| | |
|---|---|
| Teacher: | If I had the number 8 and I want to increase it what number could I make it? |
| Child 1 | I wouldn't have a clue |
| Child 2: | 19 |
| Teacher: | How much did I increase it by and how do you work it out? |
| Child 2 | 7 No |
| Child 3: | 11 adding on my fingers [demonstrated by counting on from 8] |
| Child 4: | I don't understand what you mean by increase and decrease |

**Students' generalising categories**

After phase 1 and phase 3, students were asked to record the generalisations they had discussed as a class in their own words. An examination of the responses indicated that the generalisations expressed by the students fell into six broad categories. Each category represents less sophisticated responses, with the most sophisticated response (Category 1) including a statement about the relationship between the changes made to the parts and how this relationship related to the whole. The following section describes each of the categories with a typical response for each model.

Category 1 – Increase and decrease each part by the same amount and the whole remains the same.

An example of a student's response for the length model was *When you cut copt and put the same amount of nopt it still equals lopt* and for the number model *it is decreased and increased by the same number so it stays the same number.*

Category 2 – Increase and decrease each part by the same amount.

A typical response for the length model was *You increase by the same you decrease,* and for the number model *Increase one number and add the same number that you took away and add it to the other number.*

Category 3 – Increase and decrease the parts so the whole remains the same

A typical response for the length model was *Cut a bit off copt and add a bit to nopt to make it equal to lopt,* and for the number model *You get a new number and the same sum.*

Category 4 – Increase and decrease the parts

An example of a student's response for the length model was *If you cut some from copt* *you have to add more to nopt,* and for the number model *You take one number down and one number up.*

Category 5 – Partial response (e.g., Increase or decrease one of the parts)

Category 6 – No response

The students appeared to be more at ease generalising with respect to length than generalising with number. Table 1 summarises the number of response for each of the categories in each of the models.

**Table 1:**

*Frequency of responses for each category for each model.*

| | Category | Length model | Number model |
|---|---|---|---|
| 1 | *(Increase and decrease each part by the same amount and the whole remains the same)* | 4 | 1 |
| 2 | *(Increase or decrease each part by the same amount)* | 10 | 10 |
| 3 | *(Increase and decrease the parts and the whole remains the same)* | 6 | 10 |
| 4 | *(Increase and decrease the parts)* | 10 | 9 |
| 5 | *(Increase* or *decrease one of the parts)* | 13 | 5 |
| 6 | *(No response)* | 2 | 10 |

While the trends in the responses appear to be similar, it is worth noting that, for the length model, four students gave the highest level of response and only two students failed to make any response at all. This is particularly pertinent given that the class discussion with regard to the generalization with the length model occurred before discussion about the number model. Also explicit links were made between the two models throughout the lesson and most of the lesson focused on exploring the addition compensation rule with the number model.

At the conclusion of the lesson, students were brought together for a short discussion with regard to how the generalization in the length model was similar to the generalization in the number model. The students' responses appeared to show they understood the similarity between the two models as the following discussion in one classroom shows:

Bernice: As one goes up 5 the other one has to go down by 5.
Teacher: Is that similar to the pieces of paper?
Charles: Yes as one increases the other decreases
Teacher: You have forgotten an important word. What is the important word?
Nick: By the same amount.
Sam: As one goes up the other goes down by the same amount.

## DISCUSSION AND CONCLUSION

While the aim of the lesson was to investigate instruction that assists young students recognize the addition compensation rule, it also aimed to investigate how different models assist young students extract the underlying structure of this generalization. We found that children had difficulty because of language, numbers and understanding of basic mathematics concepts such as equals.

Many children struggled with expressing the generalization in their own words, even when the generalization simply involved lengths of paper. They not only seemed to lack the mathematical vocabulary to hold conversations about mathematical ideas but also appeared unused to participating in these types of conversations. In one classroom, many students held a view of addition and subtraction that did not assist them to explore ideas; they could not see addition and subtraction as a process of change, they could only see it as a process that produced a product, the answer. It seems that not only do young students possess narrow understandings of equals (Warren, 2003) but also of the operations themselves. The impact this has on reaching arithmetic generalizations needs further investigation.

Before the lesson, we had conjectured that young students would have less difficulties in conversing about patterns without number than those with number. The results appear to support this claim. The length model did seem to allow young students to more effectively focus on the underlying concept of arithmetic compensation, thus supporting Davydov's (1975) claim. While we did not strictly follow his belief that we should begin mathematics in a numberless world, initially building up the structure of mathematics using quantitatively different models, the use of his theories in conjunction with the development of number appears to be advantageous. In addition, the representations appeared to influence students' ability to generalize. For example, the random vertical pattern in the tables generated during the lesson certainly assisted in focusing students' attention on looking for patterns across the table rather than down the table. In no instances were there any students who were not looking for the horizontal patterns.

From this preliminary research, it seems that exploring patterns in a number world increases the cognitive load for most students because, as they looked for patterns, they had to continually compute answers, and for some this went beyond their capabilities. This was particularly pertinent as the numbers became larger, with many students failing to participate in these explorations even though they had the use of a calculator to assist them. For some students, the activity had simply gone beyond the model that they commonly used for such activities, their fingers.

This results in a dilemma when exploring mathematical generalizations with numbers: can we focus on small numbers thus reducing the cognitive load or do we need to move into large numbers to that students are forced towards the generalisation in order to reduce the level of computation required? It was only when the lesson moved into large numbers that some students began to use the generalisation to find number combinations (e.g., add 3 to one number and take 3 from the other).

The use of the length model does not have these difficulties. While attempts were made to make explicit links between the underlying generalization in the length model and the number model, the effectiveness of this phase of the lesson needs further exploration. The conversation with one class seemed to support the notion that many students had effectively made the transfer, but only a few students participated in this conversation. Thus, while the use of quantitative models in conjunction with number models appears to assist the generalization process, more research is needed on the difficulties of transferring between the models and on the instruction and activities that assist with making these links.

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