EMPOWERING ANDREA TO HELP YEAR 5 STUDENTS CONSTRUCT FRACTION UNDERSTANDING[1]

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This paper provides a glimpse into the positive effect on student learning as a result of empowering a classroom teacher of 20 years (Andrea) with subject matter knowledge relevant to developing fraction understanding. Having a facility with fractions is essential for life skills in any society, whether metricated or non-metricated, and yet students the world over are failing in this aspect of mathematics (Queensland Studies Authority, 2002; TIMSS, 1997). Understanding fractions requires comprehension and coordination of several powerful mathematical processes (e.g., unitising, reunitising, and multiplicative relationships) (Baturo, 1997, 2000). While this paper will report on student learning outcomes, its major focus is to tell Andrea's story and from this to draw implications for pre-service education and teaching.

INTRODUCTION

Andrea, who was teaching in a small rural town in Queensland, was a brilliant practitioner. Using Askew, Brown, Denvir, and Rhodes's (2002), parameters for evaluating classroom interactions; I rated her at the highest level in each of the four parameters (Tasks, Tools, Talk, Expectations and Norms). Andrea had developed a truly remarkable community of mathematics inquiry where interest and engagement were often sustained in thought-provoking tasks in whole-class situations. Her principal, colleagues, and parents all attested to her teaching credibility.

I was privileged to observe an impromptu discussion which was initiated by a student who asked which was larger in value – 0.3 or 1/3. Andrea's response was to write both numbers on the board and ask: *Can we tell by just looking at the numbers*? The students replied in diverse ways; some said yes; some said no, one said $^1/_3$ is bigger because *its bits are bigger*, while another said $^1/3$ is bigger because *you can put it into ninths*. Andrea didn't tell the latter student that his thinking was inappropriate but drew, on the board, a rectangle which she partitioned into ninths, asking: *What has this been partitioned into? What can I do to show thirds?* [By doing this she showed she was looking for reunitising.] She accepted responses from students but validated them with the remainder of the class, thus setting up a debate. Finally, the class agreed on appropriate shading to show 1 third, which Andrea followed by asking: *Does this help us compare 1 third with 3 tenths*? Discussion drew from the students that "the same" (congruent) wholes would be needed to compare thirds and tenths. Only then did Andrea throw the problem back to the students to either work alone or in groups to come up with a solution.

[1]This study is part of a much larger study supported by the Department of Education, Science and Training, Canberra, Australia.

The students were "riveted" by Andrea's style of questioning and were eager participants in the discussions. No child was made to feel embarrassed by any response s/he made; in fact, Andrea made inappropriate responses the basis for her line of questioning by throwing the responses back to the class for consideration and amendment. However, even though Andrea was an excellent teacher with very good pedagogic knowledge, she was unable to help her students construct *part/whole* fraction understandings that were robust enough to apply to a variety of tasks (see Figure 1) until she, herself, understood the structural basis of the topic and gained specialist techniques that relate to such structural understanding.

BACKGROUND AND THEORY

Fundamental to the *part/whole* fraction subconstruct is the notion of *partitioning* a whole, whatever its representation, into a number of *equal* parts and composing and recomposing (i.e., *unitising* and *reunitising*) the equal parts to the initial whole. According to Kieren (1983), partitioning experiences may be as important to the development of rational number concepts as counting experiences are to the development of whole number concepts. Students, therefore, should be provided with several opportunities to partition a variety of fraction models in a variety of ways so that they come to understand that ½ (for example) always represents one of two equal pieces. Partitioning, unitising and reunitising are often the source of students' conceptual and perceptual difficulties in interpreting rational-number representations (Baturo, 1997, 1999, 2000; Behr, Harel, Lesh & Post, 1992; Kieren, 1983; Lamon, 1996; Pothier & Sawada, 1983). In particular, reunitising, the ability to change one's perception of the unit (i.e., to also see one whole partitioned into 10 equal parts as five lots of 2 parts and two lots of 5 parts), requires a flexibility of thinking that is often too difficult for some students.

Therefore, the secret to constructing a robust knowledge of the part/whole subconstruct is experience with: (a) partitioning and unitising in a variety of situations; (b) the two-way process of partitioning and unitising (i.e., constructing two-fifths when given one whole and constructing one whole when given two fifths); and (c) examples of wholes and partitioning that require reunitising (i.e., constructing three-tenths from a whole partitioned into fifths and constructing four fifths from a whole partitioned into tenths). The first of these experiences is based on developing fraction structural knowledge (Sfard, 1991) which requires teaching strategies focusing on the notion of unit/whole, partitioning the whole to form equal parts (Lamon, 1996; Pothier & Sawada, 1983), and repartitioning the whole to form smaller equal parts and to reconstruct the unit (Baturo, 2000; Behr, et al., 1992).The latter two experiences are achieved by the generic teaching processes of *reversing* (Krutetskii, 1976; RAND, 2003) and representing the concepts by both *prototypic* and *nonprototypic* models (Hershkowitz, 1989).

In this paper, I describe and analyse the changes in Andrea's knowledge and teaching practices and in the learning outcomes of her students that were a consequence of Andrea's gaining the structural knowledge and generic techniques above. This is done in order to unpack the crucial components of this teacher knowledge that had the positive effect on student outcomes. The project from which this paper emerged involved eight other mathematics-education academics and I working with eight schools across Queensland in order to identify those factors that enhance student numeracy outcomes. The schools were chosen on the willingness of three to five of their teachers to be involved in the year-long project, which was essentially a model of professional development-in-practice (i.e., in authentic classrooms). We worked with these teachers in their classrooms on topics that the teachers themselves had identified as problematic. For some teachers, this may have been related to generic pedagogical concerns (e.g., improving mathematics engagement). However, the three teachers in my school each identified aspects of mathematics that their students were failing to understand. Andrea nominated the part/whole fraction subconstruct as the area in which she felt that she was unable to teach effectively.

THE STUDY

The methodology I adopted for this study was mixed method, a combination of quantitative research on student outcomes and qualitative and interpretive research on Andrea and her class as a case study. The subjects were Andrea and her Year 5 students and the Year 5 students of a control (comparison) school that was selected to match as closely as possible the various characteristics of Andrea's class, school and community.

Data gathering methods. The data gathering methods I used were observation, interviewing and testing. To explore the process by which Andrea and her class changed their teaching and learning, I observed and videotaped the classes in which she taught fractions and talked to her regularly. To identify growth in fraction knowledge, Andrea and I administered a fraction pre-test in mid April (beginning of Term 2) and post-test in late June (beginning of Term 3). To determine the effects of the academic-teacher collaborations, Andrea and I administered a standardised test covering Number and Space (Part A) and Measurement (Part B), and a research-developed instrument covering Chance at the beginning and end of the year. The results from these classes were compared with those from a comparison school to identify changes in outcomes above those due to maturation.

I developed the fraction test (see Appendix) because the standardised test was not sufficiently detailed in the domain of fractions to provide insights into Andrea's students' specific cognitive difficulties. I needed an instrument that took cognisance of the major fraction concepts and processes, and the generic processes of reversing and representing with prototypic and non-prototypic models.

Fraction test analysis. Items 1, 2, 3a, 3c, 3e, 4 and 5a were designed to assess the students' ability to identify (unitise) fractions from a variety of representations of the whole (area, set, and linear models). Since the area models had been taught in Year 4 and the set and linear models were novel for these students, I expected that the students would perform worst on the set representations of the whole as many students find it difficult to hold a discrete set of objects in the mind as one whole. Items 3b, 3d, 3f, and 5b were designed to assess the students' ability to reunitise fractions represented by area, set, and linear models. Items 3b and 5a required the repartitioning of the given equal parts whilst 3d and 3f required a recombining of a larger number of equal parts into a smaller number of equal parts.

Items 6, 7, and 8 were designed to assess the students' ability to construct the whole when given the part (i.e., reversing). All other items had provided the whole partitioned into parts. Items 6 and 7 involved area representations. Although Item 6 involved a unit fraction, this was made more difficult by including two parts to be considered as the unit part. Item 7 was a non-unit fraction and required the students to identify the unit part. Item 8 was quite difficult as it involved a set representation of a non-unit fraction.

Procedure. After the tests had been administered, I met with Andrea to plan her unit and provide her with information for this planning. The lesson plan sequences were developed collaboratively by Andrea and me and then detailed by Andrea (Baturo, Warren, & Cooper, 2003). Emphasis was placed on the need for Andrea to develop her own activities specific to the topic.

Overarching the planning sequence was my (Baturo, 1998) teaching and learning theory which encompasses entry knowledge, representational knowledge, procedural knowledge, and structural knowledge. The use of appropriate materials to help students construct mental models was incorporated and reversing activities (e.g., "This is the whole; what part/fraction is represented?" and "This is the part/fraction; construct the whole.") and nonprototypic representations were included to develop robust knowledge. It was agreed that Andrea would include activities involving 10 and 100 equal parts in order to strengthen her students' understanding of decimal fractions.

RESULTS

I analysed Andrea's students' test responses descriptively and statistically, reread the observation field notes and viewed the videotapes, and combined the data to make a rich description of Andrea's progress.

The collaboration. The first activity was the development of the fraction test to determine Andrea's students' *Entry knowledge* (Baturo, 1998). As soon as Andrea saw the developed instrument, she knew immediately the importance of the reversing and nonprototypic representations, exclaiming: *That's what I've been missing!* Thus. Both the instrument and the students responses (which were poor in significant areas – see Figure 1) became excellent springboards for discussion of the teaching sequences for developing fraction understanding. As a consequence, Andrea spent most of Term 2 almost exclusively on re-teaching the fraction concepts, focusing first on partitioning a variety of prototypic and nonprototypic wholes, and then reunitising area, set, and linear models including nonprototypic representations of the whole and the parts, as well as on reversing activities (i.e., whole→part; part→whole). However, she found that her students were unable to process set models so these were delayed until the end of the year. Her excellent pedagogy skills combined with the more sophisticated techniques and richer representations that emerged from the joint planning meant that she was able to challenge her students' understanding at a greater depth than before.

The test results. The results of the items and sub-items (see Figure 1) show that the students initially exhibited impoverished knowledge in Items 1, 3b, 3f, 5a, 5b, 6, 7, and 8; however, as Figure 1 also shows, these greatly improved by the post-test.

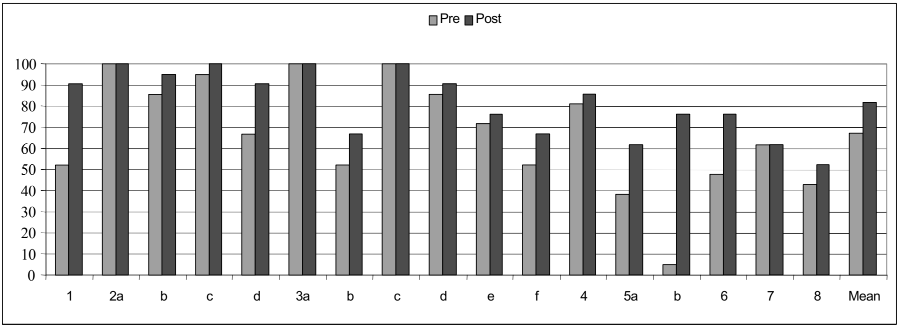

***Figure 1*. Pre and post means for Andrea's Year 5 students with respect to the fractions test.**

The increase in students' outcomes across the collaboration was also evident when I restructured the data according to the major process being assessed. Table 1 provides class and overall means (%) for the unitising items while Table 2 provides means (%) of the items related to reunitising, and reversing.

Table 1: ***Means (%) for All Unitising Items in the Fractions Cognitive Diagnostic Test***

	Unitising Items Individual Students' Means (%) and Overall Mean (%)										
	1	2a	b	c	d	3a	c	e	4	5a	Overall Mean
Pre	52	100	86	95	67	100	100	71	81	38	79
Post	91	100	95	100	91	100	100	76	86	62	90

Table 2: ***Means (%) for All Reunitising and Reversing Items in the Fractions Cognitive Diagnostic Test***

	Reunitising & Reversing Items Individual Students' Means (%) and Class Mean (%)								
	Reunitising Item					*Reversing Items*			
	3b	3d	3f	5b	Class Mean		6	7	8	Class Mean
Pre	52	86	52	05	49		48	62	43	51
Post	67	91	67	76	75		76	62	52	64

Figure 2 shows the pre and post common and decimal fraction means (taken at February and November respectively) with respect to the standardised test administered to all classes in the larger study. Pair-wise comparisons undertaken with the pre/post results revealed that the increased performance was significant for common fractions (p = 0.000) and for decimal fractions (p = 0.004). The post means also supports the supposition that the learning that occurred in Term 2 was robust for common fractions and flowed on to decimal fraction understanding.

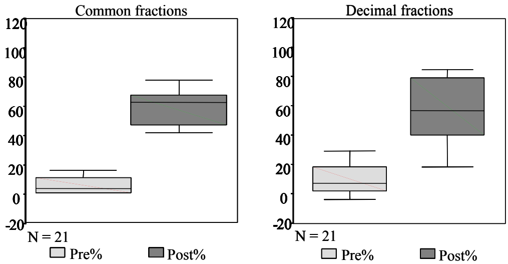

Figure 2. **Pre and post standardised test means for Andrea's Year 5 class with respect to common fractions and decimal fractions.**

With respect to relative performance on the standardised test with the comparison class, the pre-test scores were reasonably close indicating comparative starting achievement. Multivariate tests revealed a significant difference (0.05) between the classes with respect to improved performance from pre to post when pre total score was used as a covariate to control for different levels of pre performance across the schools. Pairwise comparisons revealed that Andrea's Year 5 students demonstrated greater improvement in learning than the comparison Year 5 students from pre to post overall and on Part A and Part B of the standardised test.

DISCUSSION AND CONCLUSIONS

The improvement in Andrea's class from pre- to post-tests was remarkable, more than doubling the class average. This was much higher (over double the increase) than all but one other class (also at this school) of 40 classes in the larger project (Baturo, Warren & Cooper, 2003). This result is strong evidence of the following. First, improving teachers' mathematics knowledge is a powerful method to enhance students' mathematics learning outcomes if the teachers have strong pedagogy (RAND, 2000). Second, the improvement in teachers' mathematics knowledge has to be in understanding of structural knowledge (Baturo, 1998; Sfard, 1991), the big ideas of mathematics, and in the generic teaching processes such as reversing and nonprototypic representations (Hershkowitz, 1989; Krutetskii, 1976; RAND, 2000), the big ideas of mathematics pedagogy. Third, the cognitive difficulty of topics such as the part/whole fraction subconstruct are points at which students, even with good teachers, can fail, but this can be remedied by the techniques in this paper. Fourth, the importance of Entry and a good assessment instrument cannot be overlooked – they provide the foundation for teaching as well as assessment. Last, the story of Andrea shows that good teachers are not necessarily those who know, but are those who can recognise and utilise powerful ideas when they see them. The support they need is the inclusion of these ideas in their daily teaching.

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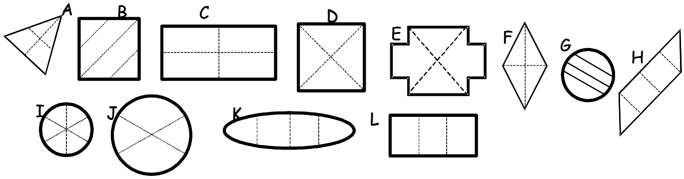
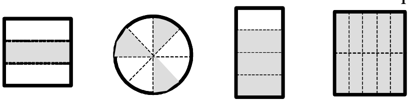
APPENDIX
C0GNITIVE DIAGNOSTIC COMMON FRACTIONS TEST

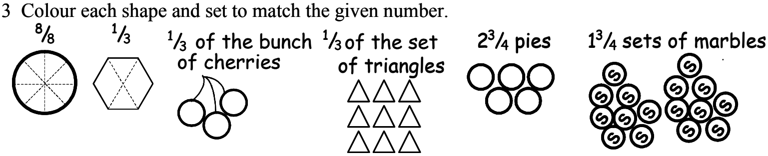
1 Tick the shapes below that have been divided into quarters.



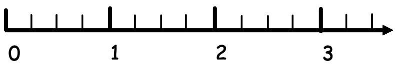
2 Write the fraction that shows how much of each shape has been shaded.



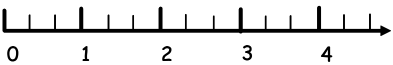
3 Colour each shape and set to match the given number.

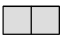


4 Show $2^1/_4$ on the number line.

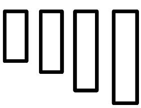


5 Write A to show where $^6/_3$ is on the number line below and write B to show where $^{11}/_6$ is.

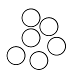


6  This is $^1/_4$ of a ribbon. Draw the whole ribbon.

7  This is $^2/_3$ of one of the rectangles below. Tick the correct rectangle.



8

This is $^3/_5$ of a set of marbles. 	Draw the set of marbles.