IMAGE - METAPHOR - DIAGRAM: VISUALISATION IN LEARNING MATHEMATICS

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The paper looks into visualisation in learning mathematics from three perspectives: It starts from a discussion what it takes to make a sign, an inscription on the blackboard, on paper or on a computer screen to an image. Here we will look into the question of 'similarity' and point to the possibility of having different perspectives on the same sign as characteristic for an image. This heuristic will be complemented by looking into inscriptions as diagrams (sensu C.S. Peirce), signs constructed and used respecting certain rules. Our main argument is that learning mathematics can be described as a continuous interplay of images and diagrams. The link between these two ways to use inscriptions is offered by metaphors, which help to structure new, maybe chaotic problem situations by means of old pieces of knowledge.

INTRODUCTION: IMAGES AND DIAGRAMS

In Didactics of Mathematics, visualisation seems to be an important issue at present: The data base 'MATHDI' (from FIZ in Karlsruhe/Germany) offers more than 300 entries (mostly in English) - and we will try to build on results from some of these publications (Presmeg, 1992, 1998; Arcavi, 2003; Kadunz, 2003).

We start from taking images as potential representations (i.e.: a not necessarily material means to speak about something), which can - by means of analogy - present a multitude of relations. They are different from symbols, which are taken as signs, which - in a specific context - only represent a single meaning, a single relation. Here, images - as analogous representations - offer the heuristical part of learning, whereas diagrams stand for the algorithmic part of learning. Images are characterised by adjectives like polyvalent and diffuse, while diagrams are should be clear and algorithmic in use. Diagrams even can be treated by machines.

The point in our argument is that these two forms of signs do complement each other in their role for learning mathematics. More specifically, most often it is the decision of the learner if s/he looks upon and uses a representation in an analogous or algorithmic vein, if s/he metaphorically describes or algorithmically transforms a given representation. We want to stress that the changing and mutually controlled use of multi-purpose representations is the most characteristic feature of visualisation in this text.

IMAGE

We start from the assumption that an image is a special, complex sign - a discussion on characteristics of this sort of signs is a good way to better understand images.

With this approach, being a sign is a major quality of an image and a theory of images is part of a theory of signs, of semiotics.

A sign is an entity, which stands for something else, which points to something else. Insofar, this 'something' is quite arbitrary, the link to its sign seems to be a convention. (in fact, it is NOT the sign, which points to something, but the person looking onto the sign who links it to the object). For being short, we will omit this complication of distinguishing between the sign and its user in the following whenever appropriate). The meaning of a sign is deeply related with its use, but: in the case of an image, this arbitrariness is restricted, because the 'something it stands for' should be recognisable. An image of a landscape is a sign of a special (or a general) landscape (as with the first landscapes in the late medieval pictures before the Renaissance). It is an image of a landscape seen by human beings. An image relates to something, it 'denotes' something (see the concept of 'denotation' by Goodman, 1976). Even if the arbitrariness of denoted objects in the case of an image is restricted because of the necessity of being recognisable, an image may denote more than one object. In addition to that, an image may point to different objects. In addition to the face, a portrait may show a building, a vase or some other everyday object in the background. Apart from the number of objects in an image, the relations between these objects may be denoted, implying an explosion of the numbers of 'objects' denoted. Medieval images of coronation ceremonies show the persons involved in sizes according to their respective importance in the ceremony. Colours and/or objects the persons have in hand offer additional information on these persons - at least to those spectators who are educated and initiated to these symbols. Consequently, the objects denoted by an image should be a restricted, not arbitrary set of objects. Normally, the collection of denoted objects is not restricted to just one object. The 'design' of an image may additionally convey a certain message on the denoted object (like its holiness or diabolic character in medieval images), the initiated spectator may discern a certain style of an image which allows to place it into a certain context (simply compare an impressionist to an expressionist picture). Under very special, 'limit' conditions, a sign may point to exactly one object - like the national flags which only stand for just one nation. For the following, we should stress and retain the ambiguity of an image.

With respect to ambiguity, we want to look into similarity, which may be a criterion to guide the attention of someone using/regarding a sign to detect the relations denoted in the image. There may be relations between parts of the image, which also exist between the objects denoted in the image. Pertinent literature calls this 'structural similarity' (Goodman, 1976, p. 231). Further informations on the concept 'image' can be found in Arnheim, 1969, Panofsky, 1982, Crary, 1992 or Waldenfels, 1994. Comments from a didactical point of view on this can be found in Kadunz (2003). The remark on structural similarity implicitly takes us to the second question of this paper, the question of metaphors, which somehow serve as a bridge to our last issue, namely diagrams. We will come back to metaphors in the final conclusion on visualisation. With respect to images, we just want to cite W.J.T. Mitchell and his book entitled „Iconology, Image, Text, Ideology“, where a material image ('picture') and its verbal image ('metaphor') are endpoints of an interval (Mitchell 1987, p. 10). For Mitchell, these two extremes are linked by 'similitude', but fall apart in terms of the way they are materialised. The image may be in front of us (on paper, on screen or on the blackboard), while the metaphor is part of the spoken language. What is typical of a metaphor, this distant relative of an image or picture?

METAPHOR

In the first chapter of his book „La métaphore vive“ (transl. to English as "The rule of metaphor", Toronto 1981), Paul Ricoeur (1975) defines the role of rhetoric following Aristotle. Contrary to the now usual definition, rhetoric should not only provide orientation for the construction of a talk, but has a role to play in controlling the validity of arguments used in the oral presentation. This control function was more and more neglected, reducing rhetoric to a means of decoration ('ornatus') of oral presentations. Somehow in contrast to this, the theory of metaphors starts from the assumption that metaphors are everywhere - and paradigms of a theory of metaphors abound. A classical definition is given by Du Marsais (ca. 1730) in his texts on language patterns: The metaphor is a pattern, which transports the meaning of a word into a meaning, which is valid only by means of a mental comparison („*Die Metapher ist eine Figur, durch welche sozusagen die eigentliche Bedeutung eines Wortes auf eine andere Bedeutung übertragen wird, die ihr nur durch die Kraft eines Vergleiches im Geiste zukommt*“; the German citation from Nöth, 2000, p. 342; transl. to English by RS). This idea of a transport is already in the word 'metaphora' (denoting something carried to somewhere else). With a metaphor, we closely link two meanings, some authors even speak of two semantic spheres. The classical theory of metaphors describes the relation between the two meanings as a relation of similarity. Aristotle himself speaks of analogy - and we have already alluded to similarity when discussing images in the section above.

Creating meaning

Describing the use of metaphors, we will concentrate on the creation of meaning and the control and revision of such meanings. Here we will draw on more recent theories of metaphors by I. Richards or M. Black. How to understand the birth and development of a new meaning of a fact when using a metaphor? With respect to this question, Richards and Black point to a special and reciprocal interaction if a context is described in an unusual way. Richard speaks of the context in terms of a *tenor* and of the unusual description in terms of a *vehicle* (Black: 'frame' and 'focus'). The metaphor "a human being is a wolf" takes the human being as the 'tenor', which is described by the metaphorical predicate 'is a wolf'. It immediately comes to mind that there is a direction in the metaphor, it is not symmetrical. Saying "the wolf is a human being" would ascribe properties to the wolf which - contrary to the usual image - would make him a creature with human (and positive) traits. Nevertheless, a metaphor also transports properties of the tenor to the vehicle. The idea that a human being is a wolf may for instance also emerge because wolves are living in groups, hence as socially organised creatures. At least Richards suggests this when a person using a metaphor takes properties from the tenor (in our example: the human being) to motivate structures in the vehicle (here: the wolf) using similarity. Reciprocally, the tenor is looked upon using properties of the vehicle. Unwanted connotations of the wolf develop into a filter to characterise human beings. Such a theory of interaction of the tenor and the vehicle looks upon the similarity as something deliberately created. Metaphors create similarities where no similarity has been before the metaphor - and this potential of synthesis is exactly how metaphors create semantic innovation, i.e.: new knowledge.

In addition to this recursive relation, there are two features typical of a metaphor: First, they resist to being paraphrased. Giving a paraphrase of a metaphor destroys what is implicitly expressed in a metaphor. The second feature is especially interesting for didacticians of mathematics: metaphors are multi-facetted, allowing a multitude of consequences from the metaphor in question. Such a multi-facettedness immediately creates a need for interpretations or simply the need to reflect on the metaphorical description. This reflection may lead to new meanings (and/or knowledge). It is exactly the confrontation of meanings which are not compatible (human being and wolf), which urge for an interpretation. "What is meant by this statement? How to understand this statement?" To cite Weinrich: "Contrary to what was traditionally thought about metaphors, metaphors do not picture real or imagined commonalities, but newly create analogies. They are the tools of demiurgs" („*dass unsere Metaphern gar nicht, wie die alte Metaphorik wahrhaben wollte, reale oder vorgedachte Gemeinsamkeiten abbilden, sondern dass sie ihre Analogien erst stiften, ihre Korrespondenzen erst schaffen und damit demiurgische Werkzeuge sind*“; Weinrich, 1976, p. 309; transl. RS). Everyday, somehow dead metaphors ('leg of a table', 'bottleneck') do not produce such a motivation, we just lexically use them as words ('arbitrary signs').

Successful new metaphors often provoke a whole bunch of implications with influences in both of the semantic fields linked by the metaphor. Metaphorically (!), one describes this as 'resonance', which induces additional similarities. Black and other authors look upon this creation of similarities as a special cognitive function of human beings (and here we are very cautious in our choice of words), which can lead to the creation of new meaning, perspectives and uses. In this construction, one overrides, maybe even violates 'mathematical logic' and uses a logic of the 'unheard' (according to the German edition of the Ricoeur text, the original French text uses the word 'impertinente', i.e.: the characterisation somehow violates the conventions normally followed). Metaphorical descriptions follow the logic of the unheard because they

- put together things which have been different and consequently never heard together before,
- provoke by being looked upon in conjunction,
- are not understood (completely) when being created or heard.

Our suggestion is that metaphors are not only semantic deviations, but should be taken as produced by us and/or by learners and as productions, which do not follow the usual rules of use. In order to understand and interpret the implications of a metaphor, to create a constructive context for them, we have to use special rules for them. We have to come to grips with the meaning imbedded in these 'unheard' descriptions. In this reflection on the embedded meaning, we need special rules for reflection -taking into account that we do not know about the type of rules we need for this. As a didactician, we assume that a learner makes use of metaphors in situations and configurations, which s/he does not fully understand. Here we are obviously not talking about routine, algorithmic procedures. For supporting the learning of mathematics, it must be helpful to allow for metaphorical descriptions if not actively demanding them. In the course of the learning, these metaphors should be stripped off their metaphorical meaning to covert them into literal descriptions fitting into the usual frame and procedures of mathematics.

In order to test the value of a metaphor, mathematics has a specific test to decide on its validity and pertinence for a solution: We suggest to use the construction and use of appropriate diagrams as a 'litmus test' for metaphors in learning mathematics. The implications of a metaphor have to show their validity when used in a diagram, the validity of metaphors is controlled by using diagrams. To put it differently, from inscriptions used as images we construct metaphors, which are controlled by diagrams, which follow the rules of mathematics.

DIAGRAM

In order to detail the task we ascribed to diagrams, we follow the definition of a diagram from semiotics offered by C.S. Peirce. For Peirce, diagrams are iconic signs, "a Diagram is an Icon of a set of rationally related objects. By rationally related, I mean that there is between them, not merely one of those relations which we know by experience, but know not how to comprehend, but one of those relations which anybody who reasons at all must have an inward acquaintance with. This is not a sufficient definition, but just now I will go no further, except that I will say that the Diagram not only represents the related correlates, but also and much more definitely represents the relations between them, as so many objects of the Icon." (see Peirce, 1906, 'PAP [Prolegomena for an Apology to Pragmatism]', NEM 4:316, c. 1906). This implies, that diagrams are created according to accepted rules of a system of representations and is used according to these rules. For instance, the grammar of a language is such a system controlling the creation of spoken and written language. In a similar way, the constructions of Euclidean geometry are diagrams insofar as they follow the conventions of this geometry (finally decided upon by the persons doing geometry). Besides other things, especially the axioms and the statements derived from them are the major conventions within geometry.

CONCLUSION

How to understand the relation of images and diagrams as they are introduced and understood above? For the use of images, we focused on their ambiguity, on the chance of creating and seeing a whole variety of relations in(to) an image. Using inscriptions as images, we concentrate on relations (and less on the things related). Concentrating on relations, we can even apply the imagistic perspective to inscriptions which - in everyday language - may not be taken as images.

To give an example, we could come up with an equation of school algebra built with a number of algebraic expressions ('terms'). In order to transpose the equation, one may bring together some terms and then try to simplify the equation. Algebra offers rules for this simplification, which are to be applied to these inscriptions, to these diagrams of algebra. On the other hand, algebra normally does not tell us which part of the equation is to simplify. The mathematician has to make a choice from (the different parts of) the diagrams in order to use the transposition rules of algebra. In this respect, the equation is looked upon as ambiguous, it is viewed as an image.

More generally, we have to explain how to see a diagram in an image, how to convert a (part of an) image into a diagram. Following Mitchell again, we make use of the opposition of visible images, of images on paper or a computer screen, which are perceived by our senses, to speech images, hence to metaphors as counterparts. From the ambiguity of an image, from a variety of relations may come up descriptions for a state of affairs presently unknown. Think about the well-known example of the "Allisons water level" (see Presmeg 1992): „Another example, which I have described elsewhere (Presmeg, 1992) involved Allison`s "water level" metaphor, accompanied by an image of a ship sailing, which reminded her that she was trying to find the key acute angle in trigonometry." (Presmeg 1998, p. 28). In this prototypic example, we can see that and how metaphors serve as 'bridges' between images and diagrams to help to cope with a situation.

One has to mention that giving a metaphorical description of a problem is no guarantee for progress in terms of a solution. The relations seen into an image, the metaphorical diagram, which describes the problem in an innovative way, has to be controlled and evaluated if it helps for a solution. Does it offer new ways to transpose the problem? Does it respect the valid rules? Can we find relations to other diagrams?

We would like to refer to a recent paper by Arcavi on „The Role Of Visual Representation": „Given were: a) the tenth term of an arithmetic sequence (a_{10}=20) and the sum of the first ten terms (S_{10}=65). The student found the first element and the constant difference mostly relying on a visual element: arcs, which he envisioned as depicting the sum of two symmetrically situated elements in the sequence, and thus having the same value. Five such arcs add up to 65, thus one arc is 13. Therefore, the first element is 13-20=-7. Then the student looked at another visual element: the ‘jumps’, and said that since there are 9 jumps (in a sequence of 10 elements starting at -7 and ending at 20), each jump must be 3“ (Arcavi 2003, p. 237; see also figure

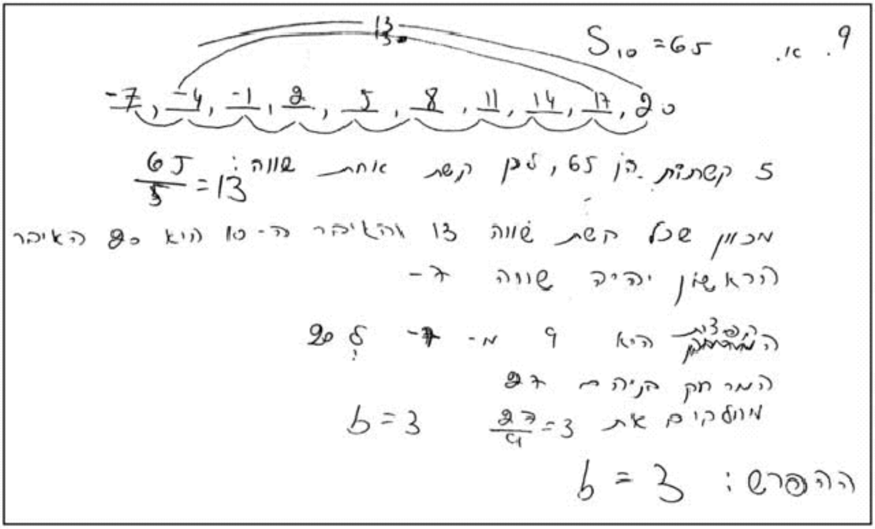

Fig. 1 (from Arcavi, 2003, p. 237)

1).

“Arcs” and “Jumps” are metaphorical means to describe the problem, which generate ideas to solve the problem and are controlled by algebraic rules. In the first step, the student obviously heavily relies on the "arc"-metaphor by linking the symmetrically situated elements, thus creating additional (in the first instance: imagistic) inscriptions (on the invention of new inscriptions see for instance diSessa&Sherin 2000). He then uses the new entities ‘arcs’ as diagrams (having 5 arcs, hence 65/5 = 13 as the sum of one ‘arc’). He applies the algebraic transposition. The diagrammatic use of his metaphor allows him to calculate the first element. He then creates the next metaphor, the "jump"-metaphor from his image to go from one element to the next, neighbouring element. This metaphor is used algebraically to find the width of the ‘jumps’. He obviously changes how he makes use of his inscription. First it is an image to generate ideas. Then it becomes a diagram, which is used according to the rules of algebra. Both perspectives luckily complement each other - linked by the heavy use of metaphors.

This is exactly how we look upon visualisation when learning mathematics: Visualisation is understood as linking images and diagrams with the help of metaphors.

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