FROM FUNCTIONS TO EQUATIONS: INTRODUCTION OF ALGEBRAIC THINKING TO 13 YEAR-OLD STUDENTS

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The various difficulties and cognitive obstacles that students face when they are introduced to algebra are well documented and described in the relevant bibliography. If it is possible, in order to avoid these difficulties, we have adopted the functional approach widening the meaning of algebraic thinking. In this paper, which is part of wider research, we concentrate on problems that are modelled by linear equation with the unknown on both sides. We investigate the advantages and disadvantages of the functional approach in the solutions of these kinds of problems. The findings from our research suggest that the functional approach indeed gives the beginners a satisfactory way of answering, while the typical solution by equation demands maturity on the part of the students and could be postponed for a later time.

INTRODUCTION

Research in teaching and learning algebra has detected a number of serious cognitive difficulties and obstacles especially to novice students, see for example Tall and Thomas (1991). One of the important themes that research has focused on is the solution of linear equations and problems related to them. Kieran (1997) and Sfard & Linchevsky (1994) have indicated difficulties related with the use and meaning of the symbol of equality, while Kieran (1985) and Kuchemann (1981) found misunderstandings about the use and meaning of letters, to name only a few. In the transition from arithmetic to algebra one of the important steps seems to be the solution of ax+b=cx+d and its variation ax+b=cx, Filloy and Rojano (1989), Herscovics and Linchevski (1994). This particular form of equation has been a subject of dispute in bibliography. For example, Filloy and Rojiano suggest that this equation demands teacher's intervention 'the didactic cut', while Herscovics and Linchevski locate their argument in the student's cognitive development. We accept the position of Pirie and Martin (1997) that rather than an inherent difficulty in the solution of linear equations, the cognitive obstacle is created by the very method, which purports to provide a logical introduction to equation solution. Our classroom experiences say that, indeed, this kind of equation puts a very heavy burden on the students.

In this paper we adopted the functional approach to algebra which widens the meaning of algebraic thinking. Then, through problems which are expressed by equations of the form ax+b=cx+d or ax+b=cx, we examine the students' solution processes by the two approaches, functional and letter-symbolic. Our goal is the investigation of the advantages and disadvantages of the functional approach in the solutions of problems, which demand the solution of equations of this kind.

A CONDENSED THEORETICAL FRAMEWORK

According to Lins (1992, p.12), to think algebraically is: (1) to think arithmetically, which means modelling in numbers, and (2) to think internally, which means reference only to the operations and equality relations, in other words solutions in the boundaries of the semantic field of numbers and arithmetical operations, and (3) to think analytically, which means what is unknown has to be treated as known. A central notion in this view is the intention to make the shift from the situational context to the mathematical context. Although this view is quite serious, it can be considered as a base of the traditional way of teaching algebra because, in the educational praxis, this approach is prone to well-known manipulation of symbols according to fixed rules.

Recently approaches have been developed in algebra that broaden the meaning of algebraic thinking. One of these approaches is the functional one, see for example Kieran (1996), Kieran et al (1996), (Yerushalmy (2000). A functional approach assumes the function to be a central concept around which school algebra can be meaningfully organized. This means that representations of relationships can be expressed in modes suitable for functions and that the letter-symbolic expressions are one of these modes. Thus, algebraic thinking can be defined as the use of any of a variety of representations in order to handle quantitative situations in a relational way, Kieran (1996, p.275). This approach can be used in two ways, namely as a cognitive support for introducing and sustaining traditional discourse of school algebra and as having its own value. In our approach we use both of them. So the research question of the paper can be reformulated as follows: Does the functional approach sustain a problem solving process in problems modelled by equation with the unknown on both sides? What are the connections between this approach and the letter-symbolic solution processes? What are the advantages and disadvantages of each one?

THE RESEARCH DESIGN

In the Greek curriculum for the second class of 13-year-old students in junior high school, equations precede the functions and in the students' textbook they are found in two different chapters. The solution of equations $ax+b=c$ and $ax+b=cx+d$ is presented in a typical way, concentrating on symbol manipulations, while in functions the linear one of the form $y=ax+b$ is the main subject. We point out that the equation chapter contains, as a final paragraph, simple inequalities. For this reason investigating the solution of inequalities was another goal of our approach, but we will not make any reference to this subject in this paper.

In order to investigate the questions above, we developed a course consisting of 26 lessons of 45 minutes each, four lessons per week. This course replaced the course on equations and the one on functions. The functional orientation enabled us to connect various problem situations to graphs, tables and letter-symbolic representations as well as to connect these representations to the notion of equation. At the beginning of the solution of a problem, attention was paid to the graphic representation of it, where x was seen as a variable rather than an unknown quantity. In this way the symbols as letters, lines or tables, acquired a meaning from the situational context of the problem. So, problems which traditionally could be answered only by the solution of an equation were now treated in many ways: By trial and error working on a table, by the graphic representation or by an equation. During the course one PC and a video projector were available to the teacher. The PC was used to provide graphs and to develop a class discussion on the qualitative aspects of the tasks. At the end of the course, one and half months later, a post-test was given to the class, considered from then on as the experimental group, as well as to a second class, considered the control group, in which the teaching of equations followed the textbook. Then, a number of interviews were implemented by the third author. Eight students from the experimental group participated in five interviews each, which covered all the subjects of the course. Moreover, two students from the control group participated in the same interviews. The subject of one of these interviews was a break-even task which was modelled by the equation $ax+b=cx$. The qualitative elaboration and comparison of the post-test results between the two groups as well as the interviews on the break-even task were the data we used for the present research. In the following we will focus our attention mainly on the break-even task, presenting extracts from two interviews, one from the experimental group and one from the control group.

THE BREAK-EVEN TASK

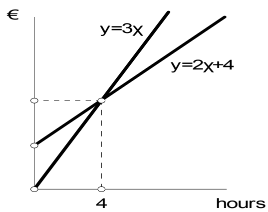
Mr. Georgiou goes by car every work day to the centre of Athens, where his office is located. Nearby there are two car parks. The first demands 4 euros to enter and 2 euros per hour. The second demands only 3 euros per hour. Mr Georgiou does not have a regular timetable. So, his choice about where he parks his car depends on how many hours he will stay at his office. Questions: 1. Express the amount of money as a function of time for both car parks 2. For how many hours can he park his car and pay the same amount of money at each car park?

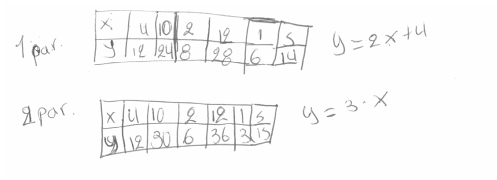
Background to the first interview

During the course the students developed experience in constructing graphs and solving problems through qualitative understanding of the given situational tasks. The solutions came from the work on graphs, with an accurate coordinate system which was distributed by the teacher, after a class discussion. In general, we used three ways to move from graphs to equations, according to the three ways of representations:

First, giving a specific value of the dependent variable on a table and asking for the associate value of the independent variable x. Second, from the graph of y=ax+b, asking for the value of the independent variable x when y had a specific value. Third, equating the letter-symbolic representations of two lines, in order to find the coordinates of their point of intersection algebraically. This last case was presented in class through the ‘renting a car’, a problem similar to the break even task. The final part of the course was devoted to the formal solutions of linear equations. In order to achieve the conceptual understanding of the steps to the solution, the teacher persisted in the justification of every step, for example he used the metaphor of balance for moving e.g. a number from one part of the equation to the other, the role of distributive low etc.

First interview, Helen (an average performer in mathematics): After the teacher presented the problem, she constructed tables according to the problem descriptions. From this work she gradually conceived the mechanism through which she developed the functions that represented the money to be paid in the two car parks, which are y=2x+4 and y=3x. Moreover, by randomly choosing x=4 she found the correct answer, see her tables below. Then the interviewer encouraged her to work in another way. Helen constructed the graphs of the two functions.





Helen: Here…at the point of intersection…well…it does not matter in which car park he will park his car, he will pay the same amount of money.

I (interviewer): How much money?

H: (she constructs perpendicular lines to two axes by hand, then she finds 4 on x axis and 11.5 approximately on y axe): 11.5 hours

I: Are you sure?

H: Yes…wait a minute…what x represents…no, in four hours he will pay 11.5 euros.

I: Are you sure that here it is 11.5?

H: No because I do not have a ruler to construct the line more accurately.

I: Can you find it in a different way?

H: …precisely…yes to make calculations…if I put y equals three times four to one and y equals two times four plus four (she refers to the formulae, then she finds the correct number, 12 euros).

Up to now Helen was able to solve the problem as she applied what she had learned in the course. She developed, even in a primitive way, abilities for planning, monitoring and, to some extent, executive control. We see that Helen connects the three different representations that model the problem. In other words, she has identified the relationships which connect tables and graphic representations to the letter symbolic system. This means that she has developed meaningful understanding of the problem, which has to do with the relationships that formal configurations of symbols have to other kinds of internal representations. The situation changed rapidly when the interviewer urged her to solve the problem using an equation. In his question 'Can you solve the problem in another way?' she was unable to answer. Then the interviewer helped her.

I: Which is the unknown in the second question?

H: The hours

I: If Mr. Georgiou parks his car x hours, how much money will he pay to the first car park?

H: x equals…do we know the y?

I: Why?

H: …because it says he will pay the same money…can I do it?

I: What?

H: y=2x+4 equals to 3x or y=3x

I: What do you think, can you do it?

H: It has two unknowns

Here the interviewer decided on a second intervention:

I: Can you write it without the y?

H: …yes…yes I can (She writes 2x+4=3x)…it is an equation

Helen's difficulty with the problem reformulation in terms of an equation was not the typical case for the other students. For example, Sotiris, a low performer in mathematics, showed considerable facility in formulating the problem by an equation, as the following extract shows:

I: Can you formulate it? (the problem's equation)

S: …we have 4+2x=3x

I: Can you explain it?

S: From the two functions we developed in order to calculate the money he must pay to two car parks.

For most of the experimental group students the difficulties arose when they started solving the equation. In this part we saw all the cognitive difficulties the bibliography has detected in the solution of equations. For example, many students were in a state of embarrassment in cases such as: manipulation of rational numbers, about the terms of an equation, the coefficient of the unknown, the use of terminology, in explaining the steps of their solution, to 'see' the best way of solving the equation, to conceive the zero (0) as an ordinary number.

Background to the second interview

The teaching of equations in the control group followed the students' textbook. Although the chapter uses problems mainly of a 'real' context as a starting point, the course quickly turns to the training of typical solutions of equations. As a result, many students are able to follow the appropriate steps to find the solution, but in most cases they do not make any reference to the problem and to the meaning of the symbols. On the other hand, the cognitive difficulties did not 'disappear' from this kind of course, as we see from the extract below.

Second interview, Sonia (an average performer in mathematics): At first she could not answer the problem. Then the interviewer suggested constructing tables, through which she gradually found the appropriate functions. The interviewer gave her the graphs of the two functions and asked her to answer the second question on her own. Then she formulated the equation 2x+4=3x which she solved correctly. During the solution process she justified her steps by practical rules such as: 'When I change sides in an equation I change the sign' or 'I separate the unknown from known quantities because we cannot add numbers and letters'. When the interviewer asked her if she could solve the problem in another way she was unable to give an answer. Then, the interviewer presented a rectangle whose two-adjacent sides have lengths of 5 and x+6. He asked Sonia to find an expression of the perimeter. Sonia wrote x+6+x+6+5+5.

I: Can you write it in a simpler form? (Se writes $6x^2 + 5^2$)

I: How much is 5^2?

S: …oh… (then she writes 12x+10)

I: When you were trying to solve the equation, in the previous problem, you said that we cannot add letters and numbers, but here what are you doing?

S: ...yes…(she writes correctly 2x+22)

I: Can we write this as 22+2x? (the interviewer's intention was the distributive low)

S: I think …yes… (then she writes 24x).

This extract shows us that teaching the algorithm of solving equations can hardly be considered as a way towards the development of algebraic thinking. On the other hand we do believe that the conceptual understanding of the solution's process of equations is very important and be the final stage of a long term study in introductory algebra.

DISCUSSION AND CONCLUSIONS

As we see from the very short evidence we have presented, the functional approach gave students a way to answer problems that are expressed by the equation with the unknown on both sides. Moreover, because of the familiarity of the situational context, the students had at their disposal a suitable referential field in which the symbols acquired meaning, see for example Gialamas et all (1999). And this meaning was used in other representations e.g. tables or equations, thus connecting, at a first level, the functional approach to letter-symbolic representation. At the same time we found indications of the development meta-cognitive processes, such as using different representations to find an answer, as Helen did in the case of 12 euros. A cognitive difficulty to this approach emerged during the first interview. We observed that the presence of the two variables x and y, can cause difficulties to some students as was the case of Helen who, as she was trying to form the equation of the problem, saw 'two unknowns' y=2x+4=y=3x. We consider this difficulty as 'natural' for beginners, but attention must be paid in such cases pointing out that y is a different 'name' for the expression 2x+4. On the other hand, it is clear to us that the solution of the equation is a difficult endeavor for beginners. Even in the case of the control group we cannot speak about conceptual understanding. These findings suggest that solving these problems using equation is not appropriate for beginners. It demands maturity, so it could be postponed for a later time, see also, for example, Yerushalmy. Until then the functional approach can help students answer these problems. On the other hand, the functional approach also has its own value because it enables the students to develop problem solving abilities such as the heuristics 'trial and error', 'draw a diagram' and, for some students, 'solve an equation'. Moreover, it fosters visual thinking, a mathematical mode especially useful today in the advent of new technology.

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