INTEGRATING THE HISTORY OF MATHEMATICS IN EDUCATIONAL PRAXIS.

An Euclidean geometry approach to the solution of motion problems

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The integration of History in the educational practice can lead to the development of a series of activities exploiting genetic "moments" of the history of Mathematics. Utilizing genetic ideas that developed during the 14th century (Merton College, N. Oresme), activities are developed and mathematical models for solving problems related to uniform motions are proposed, using the graph of velocity vs. time. The view of the covered distance as the area of the figure between the time axis and the velocity curve allows for the use of concepts and propositions of the Euclidean geometry. The use of simple geometric transformations leads to equivalent motion problems of a real context. This approach, applied to a wider range of problems, can form the basis for the introduction of basic concepts of calculus (such as integral, derivative, and their interrelation), in the context of a program of instruction in Senior High School.

INTRODUCTION

Problems in context may have an important role in the development of cognitive functions, via the mathematization of real life situations. We focus on the designing of a series of activities for solving problems related to uniform motion. For the designing of these activities we adapt the idea of the integration of the history of Mathematics into the educational praxis, exploiting genetic "moments" (Furinghetti, 1997, 2002, Furingetti & Somaglia, 1998, Tzanakis & Arcavi, 2000, Tzanakis & Thomaidis, 2000, D' Ambrosio, 2001, Katz, 1997). This perspective is consistent with a more general view that the way in which mankind developed mathematical knowledge, is also the way in which individuals should acquire mathematical knowledge (Polya, 1963, Freudental, 1973, 1991).

The designing of the activities is inspired from the study methods of motion during the later Middle Ages (14th century, Merton College, N. Oresme). This methodology, as it is historically documented, was the genetic "moment " for mathematical concepts like function, graphics representation and the integral (Clagett, 1959, 1968, Gravemeijer & Doorman, 1999).

Oresme's geometrical representation model of motion, "reconstructed" in its modern form, corresponds to the (U, t) graph of velocity vs. time. In this "holistic" graph, velocity and time exist concurrently and so does distance covered, as the area of the figure between the curve and the time axis. The view of the distance covered as an area, allows the use of concepts and propositions of Euclidean geometry. We present a mathematical model for solving problems related to motion based solely on Euclidean geometry. This point of view is based on the equivalence of area of geometric figures and mainly on the invariance of these areas under translation. Simple geometrical transformations, which may be performed on the velocity graph, lead to equivalent mathematical situations and therefore to equivalent real problems.

The use of Euclidean geometry in the solution of such problems leads to the interaction of the embodied and the proceptual mathematical worlds (Tall, 2003) and allows for the study of motion based solely on the functional approach. It is a course that may lead students, through their intuition and through mathematization, to the creation of mental models in order to solve the given problems, and through generalization and formulation to the understanding of mathematical concepts.

As an example we apply our Euclidean geometry approach to the solution of a motion problem.

THE INTEGRATION OF HISTORY IN EDUCATIONAL PRACTICE

The integration of history in didactic practice helps the students to understand that mathematics is not a fixed finalized knowledge system, but a live development process, closely connected with other branches of science (Furinghetti, 2002, Furingetti & Somaglia, 1998, Tzanakis & Arcavi, 2000, Tzanakis & Thomaidis, 2000). It helps the student to understand that mistakes, doubts, intuitive arguments, controversies, and alternative approaches to problems are not only legitimate but also an integral part of mathematics in the making (Tzanakis & Arcavi, 2000, p. 205).

We used the integration of historical data in designing the activities in such a way that history is not visualized as the main element in the classroom. This type of integration is described as,

> … a reconstruction in which history enters implicitly, a teaching sequence is suggested in which use may be made of concepts, methods and notations that appeared later than the subject under consideration, keeping always in mind that the overall didactic aim is to understand mathematics in its modern form (Tzanakis & Arcavi, 2000, p. 210).

In our research firstly, we located the genetic historical "moments" for basic mathematical concepts, like function, graph and the integral. These genetic ideas were developed during the 14th century by the Calculators (Merton College) and N. Oresme and were related to the study of motions. Secondly, we integrated these historical ideas in the development of a course consisting of activities that aim to the understanding of the basic concepts of calculus. Here we present the first part of this designing that aims to prepare the students for an intuitive approach to these concepts.

GENETIC HISTORICAL MOMENTS IN THE STUDY OF MOTION DURING THE 14TH CENTURY

The mathematicians-logicians of Merton College at Oxford (1330 – 1340), are known as Calculators. The Calculators (William Heytesbury, Richard Swineshead, John Dumbleton) studied the motion of bodies. They introduced the idea of functional relations, in an attempt to describe qualitative magnitudes (velocity, distance, time), with quantitative measurable features (Gravemeijer & Doorman, 1999). They defined several kinds of motion, proposed theorems concerning motion and proved these mathematically (Clagett, 1959). Their proofs were based on Euclidean geometry. Swineshead in " De motu" defined uniform motion as follows:

> Uniform local motion is one in which in every equal part of the time an equal distance is described. (Clagett, 1959, p. 243)

Heytesbury in "Rules for Solving Sophisms" (Regule solvendi sophismata), defined uniform accelerated motion, as follows:

> For any motion whatever is uniformly accelerated (*uniformiter intenditur)* if, in each of any equal pars of the time whatsoever, it acquires an equal increment (*latitudo)* of velocity (Clagett, 1959, p. 237).

Nicole Oresme, (1362), in "De configurationibus qualitatum" represented the variations of qualities, such as velocity, by geometrical figures. The basic idea of this representation is fairly simple: Geometric figures may be used in order to represent the quantity of a quality. As examples of Oresme's technique, let us consider the rectangle and right triangle in figure 1. Each figure measures the quantity of some quality (velocity). Line AB in either case represents the "extension" (time) of the quality. But in addition to extension, the "intensity" of the quality from point to point in the base line AB has to be represented. This, Oresme did by erecting lines perpendicular to the base line, the length of the lines varying as the intensity varies. Thus at every point along AB there is some intensity of the quality, and the sum of all these lines is the figure representing the quality.

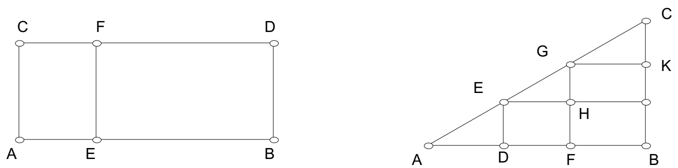


figure 1

Now the rectangle ABCD represents a uniform quality, since the lines AC, EF, BD represent the intensities of the quality at points A, E, and B (E being any point at all on AB), are equal, and thus the intensity of the quality is uniform throughout. In the case of the right triangle ABC it will be equally apparent that the lengths of the perpendicular lines representing intensities uniformly increase in length from zero at point A to BC at B, according to Merton College's definition of uniformly accelerated motion (AB is the time line). It is worth pointing out that Oresme designated the limiting line CD (or AC in the case of the triangle) as the "line of summit" or the "line of intensity". This corresponds to the "curve of motion" in modern analytic geometry. Oresme understood that the areas of the figures in the case of local motion represent the distances traveled in the corresponding times represented in each case by AB (M. Clagett, 1959, 1968).

THE DESIGNING OF THE ACTIVITIES

Tall (1996) argues that we should give greater emphasis to the visual representation of mathematical concepts. All over the world, the curricula emphasise the necessity of interconnecting the algebraic approaches with numerical and graphical ones, in the first contacts with functions and analysis. The triple: "numeric-graphic-algebraic" has become emblematic of different countries's projects. (Robert & Speer, 2001). Didactic research tends to offer arguments to support such a strategy by showing the role played in conceptualisation processes by flexibility between different semiotic registers of representations (Duval, 1995 as cited in Robert & Speer, 2001).

Taking this strategy into account and utilizing the described genetic historical ideas, we integrate the History of Mathematics in designing a range of problems related to uniform motion. For the solution of these problems we encourage the students to use the familiar Cartesian axes system and to focus on the graph of velocity vs. time. We encourage them to investigate the relation between “the area below the graph” and the distance covered in a period of time. We argue that the students, through guided reinvention (Freudental, 1991), can come to grips with the basic idea that the covered distance function can be expressed as the area of the graph function. From this geometric-graphical context, which represent the motion scenario, the students are asked to shift to the algebraic context and the algebraic formulas of the velocity and position functions. According to Tall (2003), at this point there is a shift from the embodied to the proceptual mathematical world (figure 2).

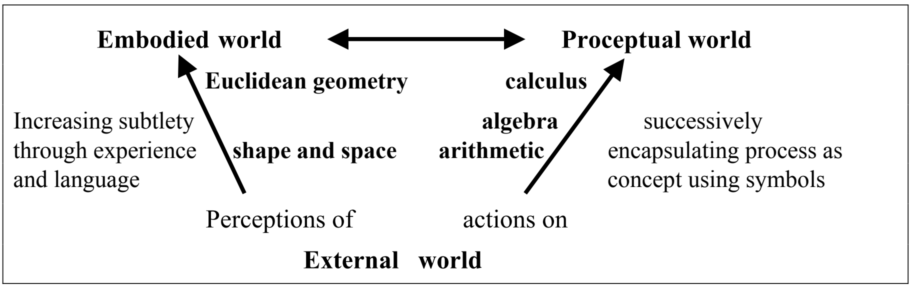


Figure 2

The covered distance of the moving body, viewed now as area, allows the use of concepts and propositions from Euclidean geometry in problem solving. Using the velocity – time graph, the students are able to solve uniform motion problems, based solely on Euclidean geometry.

We will present this solution model by an example. The solution to this problem is based on the equivalence of geometric figure areas and mainly on the invariance of these areas under translation. The geometric transformations that may be applied to the velocity graph lead to equivalent mathematical situations, and therefore to equivalent real problems. The understanding by the students that the same geometrical model can solve problems with different situational structures, can lead them to a classification of these problems.

AN EXAMPLE

Consider the following motion problem (cited in Yerushalmy and Gilead, 1999).

Problem 1.

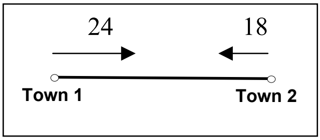
A biker traveled from town 1 to town 2 at an average speed of 24 km/hour. Arriving at town 2, she immediately turned back and traveled to town 1 at an average speed of 18 km/hour. The whole trip took 7 hours. How long was the trip in each direction?

Traditional approach

Table for the trip

	time	speed	distance
Out	t	24	24t
Back	7- t	18	18(7 – t)

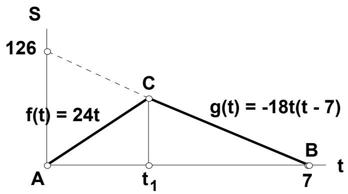
Traveling path



$$24t = 18(7 - t)$$

Figure 3. A traditional solution (cited in Yerushalmy and Gilead, 1999)

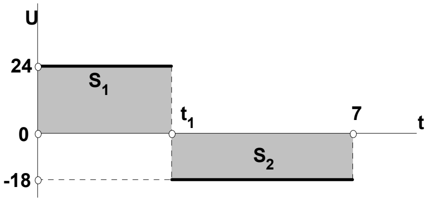
Functional approach (S, t)



$$f(t) \overset{?}{=} g(t)$$
$$24t = -18(t - 7)$$

Figure 4. A functional approach (cited in Yerushalmy and Gilead, 1999)

A "holistic" functional approach (U, t)



$$S(t) = \begin{cases} 24t, & t \in [0, t_1] \\ 24t_1 - 18(t - t_1), & t \in [t_1, 7]. \end{cases}$$

t_1 is the going out time.

$S_1 = S_2 \Rightarrow 24t_1 = \left|-18(7 - t_1)\right| \Rightarrow$

$\Rightarrow \ldots \Rightarrow t_1 = 3.$

Figure 5. A functional approach based on the "holistic" graph (U, t)

In this approach we have a shift from the embodied to the proceptual mathematical world. The formula of the position function results from the area of the rectangles and the equation for the solution of the problem is determined from the equality of the areas.

A Solution using Euclidean geometry

We give a solution to this problem, based solely on Euclidean geometry.

In figure 6, we have the representation of the problem, given that the two areas S_1, S_2 are equal ($S_1 = S_2$). Sketching two equal rectangles with area d, as shown in figure 7, we have, according to our hypothesis, that $S_1 + d = S_2 + d$. We know that the area $S_2 + d$ is the area of the rectangle with sides 7 and 18, hence is equal to 126 ($S_2 + d = 18\cdot 7 = 126$). On the other hand, $S_1 + d$ is the area of the rectangle with sides 42 (24 +18) and t_1 (the going out time). Thus, from $S_1 + d = S_2 + d$, we get $126 = 42\cdot t_1$ and consequently $t_1 = 3$.

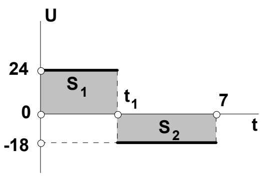


Figure 6.

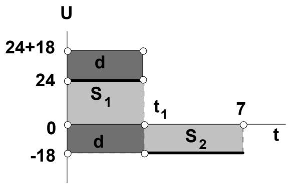


Figure 7.

Hence we gave a solution to the problem using the representative model of Oresme and simple geometric concepts.

Taking into account that the students have a good understanding of the invariance of area under translations since an early age, it is possible to lead them to equivalent real life problems, through graph transformations. For example (Figure 8):

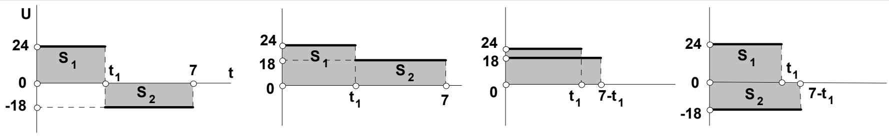


figure (a) (initial situation)

figure (b) (S_2 reflected in the t-axis)

figure (c) (S_2 reflected in the t-axis and translated by t_1 to the left)

figure (d) (S_2 translated by t_1 to the left)

Figure 8

We encourage the students to interpret the above graphs and to describe real problems of motion, corresponding to each graph. As an example, we give a real problem corresponding to figure (c).

Figure (c): Two bikers started their journey at the same time from town 1 to town 2 at an average speed 24 km/hour and 18 km / hour respectively. The time of the first biker was t_1 and the second t_2. If $t_1 + t_2$ is totally 7 hours, how long was the trip for the bikers?

The students are allowed to try several other geometric transformations and to express new equivalent real problems.

This approach applied to a wider range of uniform motion problems can lead the students to the statement that problems in which the constant velocities U_1 and U_2 are given and there is a known linear relation between the covered distances and the corresponding times (e.g. $S_2 = \kappa S_1 + \lambda$ and $t_2 = \mu t_1 + \nu$), can be solved with the same mathematical model. In fact, by employing geometric transformations, the basic unknown can be viewed as a side of a rectangle, with the area and the other side known.

Discussion

"Past never dies" said Fauvel (1991), arguing towards the integration of the history of mathematics and didactical praxis. A "past" such as Euclidean geometry, should be used appropriately in educational practice. By using simple geometric concepts, it is possible to create mathematical models for solving real problems and reach equivalent real situations via simple geometric transformations. New procedures and mathematical objects independent of the specific situation should emerge. The interconnection of the embodied and proceptual mathematical worlds (Tall, 2003), allows the deep understanding of concepts and leads to bridge the gap between informal and formal knowledge.

Finally through progressive mathematization, the need of formal mathematical thought must become clear to the students. We believe that the solution of motion problems through the velocity - time graph, could lead to a future didactic method for the introduction of the definite integral and fundamental theorem of calculus. This is exactly what the History of mathematics teaches us. The evolution of concepts during time can inspire a chain of teaching activities with specific goals, without abolishing the essence of mathematical knowledge.

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