# PROOFS THROUGH EXPLORATION IN DYNAMIC GEOMETRY ENVIRONMENTS

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*The recent development of powerful new technologies such as dynamic geometry softwares (DGS) with drag capability has made possible the continuous variation of geometric configurations and allows one to quickly and easily investigate whether particular conjectures are true or not. Because of the inductive nature of the DGS, the experimental-theoretical gap that exists in the acquisition and justification of geometrical knowledge becomes an important pedagogical concern. In this article we discuss the implications of the development of this new software for the teaching of proof and making proof meaningful to students. We describe how three prospective primary school teachers explored problems in geometry and how their constructions and conjectures led them "see" proofs in DGS.*

## INTRODUCTION

DGS has revitalized the teaching of geometry in many countries and has made necessary a radical change to the teaching of proof (de Villiers, 1996). One of the most welcome facilities of dynamic geometry is its potential to encourage students' "research" in geometry. In such a research-type approach, students are inducted into theorem acquisition and deductive proof. Specifically, students can experiment through different dragging modalities on geometrical objects they construct, and consequently infer properties, generalities, or theorems. Because of the inductive nature of DGS, the experimental and theoretical gap that exists in the acquisition and justification of geometrical knowledge becomes an important pedagogical and epistemological issue. In this paper, we discuss the pedagogical aspects of introducing DGS into the teaching of geometrical proofs and we provide some indications of how DGS can be used to offer insight and understanding of proofs through investigation and experimentation.

## THEORETICAL FRAMEWORK AND PURPOSE OF THE STUDY

### The Gap Between Proof and Exploration

The exploration of a problem is by its nature empirical, and, at a first glance, it seems that it does not fit into the deductive character of geometrical proofs. When the empirical and inductive dimension is to be added to the pedagogical structure that is traditionally rooted in deductive logic, one has to combine these two seemingly opposite perspectives. The problem of combining inductive exploration with the deductive structure of geometrical proofs has been the subject of a number of research studies (Mariotti, 2000). The traditional teaching emphasizing that a mathematical statement is true if it can be proved, led students distinguish proof from exploratory activities. However, de Villiers (1996) and (Hanna, 2000) indicated that in actual mathematical research, mathematicians have to first convince themselves that a mathematical statement is true and then move to a formal proof. It is the conviction that something is true that drives us to seek a proof. In DGS, students can easily be convinced of the general validity of a conjecture by seeing its truth displayed on the screen while geometrical objects undergo continuous transformations (de Villiers, 1996, 2003).

A number of researchers showed that the passage from "exploratory" geometry to the deductive geometry is neither simple nor spontaneous. Hoyles and Healy (1999) indicated that exploration of geometrical concepts in a DGS environment could motivate students to explain their empirical conjectures using formal proof. They found that DGS helped students to define and identify geometrical properties and the dependencies between them, but when students worked on proofs, they abandoned the computer constructions. The latter leads to the argument that DGS may be useful only in helping students understand problems in geometry but it does not contribute to the development of their abilities in proofs, reinforcing the idea that there exists a gap between dynamic geometry and proof. This may also be the reason that some educators and researchers expressed their concerns and worries that DGS could lead to the "further dilution of the role of proof in the high school geometry" (Chazan, 1993, p. 359). However, the main discussion of recent research, and the main purpose of the present study were to find out ways of effectively utilized DGS to introduce proof as a meaningful activity to students. This can be achieved by reconceptualizing the functions of proofs.

**The Functions of Proof**

Proof performs a wide range of functions in mathematical practice, which are reflected to some extent in the mathematics curricula. The NCTM Standards (2000) emphasized in a special section on reasoning and proof, the investigations, conjectures, evaluation of arguments and the use of various methods of proofs. From NCTM's document it is assumed that proof is not only understood in the traditional rigid and absolute way, but it also embraces many other functions. Hanna (2000), based on recent research on proof, provided a list of the functions of proof and proving: verification, explanation, systematization, discovery, communication, construction, exploration, and incorporation. She also considered verification and explanation as the fundamental functions of proofs, because they comprise the product of the long historical development of mathematical thought. Verification refers to the truth of a statement while explanation provides insight into why this statement is true.

Traditionally, the function of proof has been seen almost exclusively in terms of the verification of the correctness of mathematical statement. The idea is that proof is used mainly to remove either personal doubt and/or those of others; an idea which has one-sidely dominated teaching practice and most of the research on the teaching of proof. However, de Villiers (2003) proposed other important functions such as explanation, discovery, intellectual challenge and systematization, which in some situations are of greater importance to mathematicians than that of mere verification.

Edwards (1997) defined the term "conceptual territory before proof" by indicating that conjecturing, verification, exploration and explanation constitute the necessary elements that precede formal proofs. The conceptual territory provides the arena for the construction of intuitive ideas that may subsequently be tested and confirmed through formal methods, and it is the basis for a richer understanding of a proof. This approach reflects the "quasi-empirical" view of mathematics in which understanding proceeds from students' own conjectures and verifications to formal proofs (Chazan, 1993). Simpson (1995) differentiated between "proof through logic", which emphasized the deductive nature of proof, and "proof through reasoning", which involved most of the functions of proofs as were listed by Hanna (2000). Proof through reasoning is accessible to a greater proportion of students, because it is closer to the learning style of students, it makes mathematics more useful and enjoyable, and it reflects the quasi-empirical view of mathematics and the process adopted by mathematicians when they invent mathematics (Simpson, 1995).

## The functions of proofs and DGS

The availability in the classroom of DGS gave a new impetus on the teaching of geometry based on students' investigations and explorations. This does not mean that proof is replaced by explorations. On the contrary, exploration is not inconsistent with the view of mathematics as an analytic science or with the central role of proof. Polya (1957) emphasized the connection between deductive reasoning with exploration. He pointed out that solving a problem amounts to finding the connection between the data and the unknown, and to do it, one must use a kind of reasoning based on deduction. In the DGS environment students acquire understanding through verifying their conjectures and in turn this understanding solicits further curiosity to explain why a particular result is true. Students working in the DGS environment are able to produce numerous corresponding configurations easily and rapidly, and thereby they have no need for further conviction/verification (Holzl, 2001). Although students may exhibit no further need for conviction in such situations, it is important for teachers to challenge them by asking why they think a particular result is true (De Villiers, 2003, 1996). Students quickly admit that inductive verification merely confirms and the "why" questions urge them to view deductive arguments as an attempt for explanation, rather than verification (Holzl, 2001). Thus, the challenge of educators is to convey clearly to the students the interplay of deduction and experimentation and the relationship between mathematics and the real world (Hanna, 2000).

## THE STUDY

This article presents an account of the thinking exhibited by three prospective primary school teachers while attempting to answer proof problems. It is conjectured that DGS provides an appropriate context where the significance of proof may be un-forcefully recognized. To this end, the development of "appropriate" tasks was necessary. By "appropriate" we mean tasks where proof may be providing insight-illumination into why a result, which can be seen on the screen, is true. Open-ended problems seemed as more "appropriate" for two main reasons: (a) statements are short and do not suggest any particular solution methods, and (b) questions are different from traditional closed expressions such as "prove that …", which present students with an already established result (Jones, 2000). Open-ended problems give students the opportunity to engage in a process, which utilizes a whole range of proof functions: exploring a situation, making conjectures, validating conjectures and proving them. The implicit assumption is that during this process students will not have to prove something that they are presented with and do not understand, but something that they have discovered, validated and is meaningful to them. The participants in this study have been asked to work on the following open-ended problem suggested by de Villiers (1996):

Problem: Construct a kite and connect the midpoints of the adjacent sides to form an inscribed quadrilateral. What do you observe in regard to this inscribed quadrilateral? Write down your conjecture. Can you explain why your conjecture is true? Change your kite into a concave kite. Does your conjecture still hold?

After the exploration of this problem, students were engaged in proving similar geometrical theorems. The aim of these additional problems was for students to utilize the proving process in systematizing and generalizing their results.

### Students' Proofs

Three prospective primary school teachers with prior experience in dynamic geometry participated in this study. These students had attended a course on the integration of computers in elementary school mathematics, and thus they had a basic understanding of Sketchpad's drawing, menus, and construction features.

Interviewees participated on a voluntary basis and were interviewed while working on the problem. The interviews were conducted in the mathematics laboratory equipped with computers loaded with the Greek version of the Geometer's Sketchpad. The setting was informal with students being able to analyze and build geometric constructions that they thought would help them solve the problems without any time constrains being set. Unstructured interviews were used to collect the data.

In the following, we analyze students' strategies and try to underline the different aspects and functions of proof. The discussion of students' solutions to both problems is organized around three phases: (a) the phase before proof, (b) the proof phase, and (c) the phase of intellectual challenge of extending proof to similar problems.

**The phase before proof**

At this phase students explored the problem through constructing the kite and rearranging the constructed figure by dragging it in different directions. This exploration led students to form their own conjectures about the solution of the problem by visualizing the transformations that resulted by the dragging facilities of the software.

Figure 1 shows the way in which students constructed the kite and consequently the inscribed quadrilateral. Two of the students constructed the kite using the property of perpendicularity of its diagonals (see Figure 1a), while the third one used the property of equal adjacent sides by firstly constructing a triangle and then reflecting it on one of its sides (see Figure 1b). All students managed to find the midpoints of the adjacent sides and connected them with line segments using the appropriate functions provided by the software. They conjectured that the inscribed quadrilateral might be a rectangle and confirmed their conjecture by dragging the vertices of the kite to new positions. Students also realized that their conjectures hold also in the case of the concave kites. All the students evaluated their mathematical conjectures not only visually but also numerically by measuring the sides and angles of the inscribed quadrilateral, confirming that it was a rectangle, and thus verified their conjecture. It is also important to note that these students used the measuring tools for slope to show that the opposite sides of the inscribed shape were parallel. Furthermore, they noticed that the diagonals of the kite were also parallel to the sides of the inscribed shape.

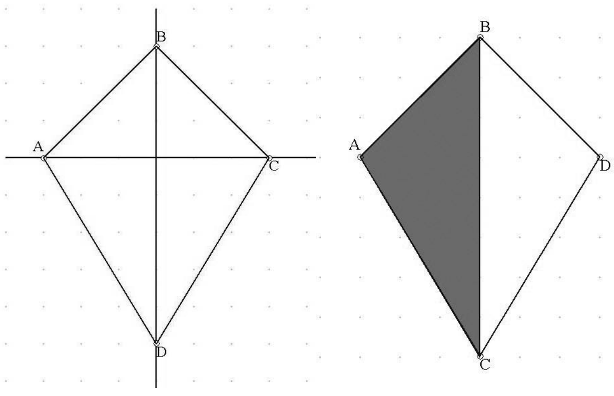


Figure 1a Figure 1b

**Figure 1: The construction of kite**

## The proof phase

The exploration of the problem as it was done in the "phase before proof" led students to become convinced about the validity of their conjecture. This conviction was achieved solely by the use of the dynamic geometry environment. During the "proof phase" the role of proof is not to convince or remove individual or social doubt about a proposition but primarily to find ways to explain why a certain result that can be seen on the screen is true (Jones, 2000). One of the students in this study showed no further need for conviction that the inscribed quadrilateral was a rectangle, while the other two students felt the need to explain why they thought this particular result was true. These two students admitted that the inductive verification they provided for the mathematical statement was not satisfactory in the sense that the inductive process was not a consequence of other familiar results. Furthermore, they proceeded to view a deductive argument as an attempt for explanation, rather than for verification.

At this phase, the DGS enabled students to pass from "exploratory" geometry to deductive geometry, bridging in this way the gap between dynamic geometry and proof. Specifically, the two students, who successfully solved the problem, based on the measurements they made earlier on in the exploration phase (the pre-proof phase), defined and identified the geometrical properties and the dependencies between them, and provided a deductive proof of the problem. In fact, they realized from their measurements that EF, and HG are equal to ½ AC (see Figure 2). This directed them in what they needed to look for in their geometry books, where they found the respective theorem. Based on this property they showed that EF is equal and parallel to HG as well as EH is equal and parallel to FG, and therefore EFGH is a parallelogram. The next step was to prove that the parallelogram was a rectangle, i.e., one at least of the angles of the parallelogram was a right angle. Based on the property of the perpendicularity of the diagonals of the kite, students observed that since $BD \perp AC$, then $EF \perp EH$, which implies that EFGH is a rectangle. (The dragging facility of the software enabled students to conceive that their explanations hold even in the case of concave kites).

## The phase of intellectual challenge of extending proof to similar problems

In this phase we discussed two categories of problems: (a) problems that have a similar context to the kite problem, and (b) problems that require the same type of reasoning. The purpose of the problems in the first category was to help students generalize their finding from the kite problem to quadrilaterals of various types. To this end, the three students tried to systematize their experimentations by investigating first the more familiar quadrilaterals such as parallelograms, rectangles, rhombuses, squares, rectangles and then they proved, using the same explanations as they did in the kite problem, that in any quadrilateral the shape resulting from the midpoints of its sides is always a parallelogram. The purpose of the second category of problems was to ensure that students could easily transfer the proving process to problems with different structure.

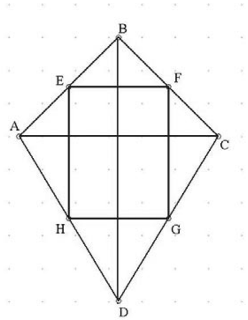


**Figure 2: The proof that the inscribed quadrilateral is rectangle**

## CONCLUSIONS

In this paper we tried to show some of the ways in which DGS can provide not only data to confirm or reject a conjecture, but ideas that can lead to a proof. To this end, the results of the study were presented in three phases: the phase preceding proof, the proof phase, and the phase of intellectual challenge of extending proof to similar problems.

The phase preceding proof is quite necessary for students to understand the problem based on their own intellectual efforts. In the kite problem students encompassed their informal reasoning and argumentation that came into play when students worked from their own investigations (Edwards, 1997). To construct the kite, which was a challenge by itself, students first investigated its properties and then tried to apply them on the computer screen. The graphing and validating capabilities of DGS enabled students to explore the problem and make mathematical conjectures. In turn, students checked specific cases of kites, using the dragging facility of the software, to see if their conjecture holds true, i.e., the shape formed by connecting the midpoints of adjacent sides of a kite is always a parallelogram. In other words, the phase preceding proof helped students to build up empirical evidence for the plausibility of their conjectures.

A number of research studies indicated that engaging students in the phase preceding proof did not necessarily lead them to an awareness of the need for proof (Chazan, 1993; Edwards, 1997). On the contrary, in the present study, we found that DGS and appropriate questions prompted or motivated students to seek justifications for their conjectures. Two of the three students in this study justified their conjectures for the kite problem based on the screen outputs. In addition, students in the study did not support that their experiments and measurements were sufficient to support a geometrical statement. Measurements functioned as a means for finding explanations and a means for gathering information for justifying their results. The relations between the measurements in conjunction with the invariant properties of the shapes functioned as students' hints into explaining their conjectures. Measurements also provided students with specific examples that formed the ground for further conjectures and generalizations. It is in this area that the computer contributed to students' attempts toward proof and bridging the gap between inductive explorations and deductive reasoning. This became more apparent during the phase of intellectual challenge of extending proof to similar problems. During the last phase, which was not adequately presented due to space limitations, students felt a strong desire for explaining their conjectures and understanding how one conclusion is a consequence of other familiar ideas, results or theorems. Students found it quite satisfactory to view a deductive argument as an attempt for explanation rather than for verification (de Villiers, 2003).

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