A NEW PRACTICE EVOLVING IN LEARNING MATHEMATICS: DIFFERENCES IN STUDENTS' WRITTEN RECORDS WITH CAS

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Students who learn mathematics with CAS calculators are likely to develop new practices for doing and recording mathematics. Students discussed in this paper were able to use CAS calculators in examinations, making their own decisions about what to document as written records for solutions to problems. A comparison of some features of written records produced by these students, with an achievement matched random sample of students with only graphics calculators, gives insight into the new practice which is emerging. Students who had learned with CAS wrote generally shorter answers, used more ordinary words and used function notation more frequently but they did not over-use non-standard calculator syntax.

INTRODUCTION

When students learn mathematics with CAS calculators and can use it in examinations, they are likely to use a combination of CAS and pen-and-paper techniques, making decisions about which is more efficient based on previous experiences and personal competency with CAS and pen-and-paper techniques. Because intermediate steps in routine procedures carried out with technology are not available for inspection, students who use CAS cannot provide the reader (including an unknown examiner in a high-stakes examination) with the traditional form of written record of solution. A new mathematical practice is therefore likely to evolve in this situation. This paper reports a study of this evolving practice, by documenting four features of the written records provided in a high-stakes university entrance and final school examination, by a group of students using CAS and comparing them to a matched group of students not using CAS.

There is little relevant literature. The strong literature related to argumentation and communication in mathematics classrooms (for example, Yackel, 2001; Krummheuer, 1995) has a focus on how students communicate mathematical thinking during classroom interactions, but is overwhelmingly concerned with verbal communication, dialogue and interaction. Our concern here is different because it deals with written communication of mathematical thinking and also because it is concerned, not with the process that students went through to solve problems, but with the end product, the written record, which is used to communicate a mathematical solution, in this case to an examiner. Literature dealing with the effects of CAS on teaching, learning and examinations has also not considered written records. An important thread in this literature (see for example, Artigue, 2002; Guin and Trouche, 1999; Pierce and Stacey, to appear) deals with the development of effective use of CAS in the classroom, but not how students communicate mathematical thinking once CAS is used to solve problems. Similarly the literature related to assessment with CAS (see for example, Flynn, 2003; Kokol-Voljc, 2000) considers required changes to assessment items when intermediate steps are not available, but not how the responses should be written.

This paper will analyse four features of written records for selected problems in the 2002 Year 12 externally set and marked Mathematics examinations in Victoria, Australia. The written records were produced by two cohorts of students, one that learned mathematics with CAS calculators and one that learned mathematics with graphics calculators. The CAS students were from three schools offering *Mathematical Methods (CAS),* a new subject (Victorian Curriculum and Assessment Authority, 2001) offered in Year 12 for the first time. The students had used CAS in both 2001 and 2002. Further details of their program and learning are available from the project web-site (www.edfac.unimelb.edu.au/DSME/CAS-CAT) and Stacey (2003). The other students were undertaking the standard subject *Mathematical Methods* (VCAA, 1999). Examinations in both subjects were externally set and graded, with a number of common questions.

Ball (submitted), a revised version of her 2003 CAME paper (see http://ltsn.mathstore.ac.uk/came), has previously reported differences in the written records of the CAS and non-CAS students for one of the common examination questions (Question 1b). She found that CAS written records tended to be shorter on average than non-CAS written records and that CAS written records contained more words than non-CAS written records. For example, Question 1b involved solution of two simultaneous equations and more than 40% of CAS written records (n=78) contained the word 'solve' while only one non-CAS written record (n=78) included this word. CAS written records also contained more function notation than non-CAS written records and there was evidence of some non-standard notation that could be directly linked to a CAS entry. From the analysis of Question 1b, it appeared that CAS students were developing a practice for writing mathematical solutions that had a number of differences to the practice being observed in the work of the non-CAS students. This paper will consider two more questions from the same examination and carry out a similar analysis to investigate whether the differences in written records observed in Question 1b are apparent in other questions.

CAS AND NON-CAS STUDENTS

During 2002 the students undertaking year 12 *Mathematical Methods (CAS)* learned mathematics with a TI89, HP40G or CASIO FX 2.0 CAS calculator and use of CAS was unrestricted in the final examination. These students are referred to as CAS students and their scripts as CAS scripts (see Table 1). Note that this is regardless of whether the student actually used CAS in the solution being analysed. Mathematical Methods students learned mathematics with a graphics calculator and could use it without restriction in the examination. These students are referred to as non-CAS students and their scripts as non-CAS scripts. Differences are summarized in Table 1.

	CAS student	Non-CAS student
Examination script	CAS script	Non-CAS script
Subject studied	Mathematical Methods (CAS)	Mathematical Methods
Technology used to learn mathematics	CAS (TI89, HP 40G, Casio FX 2.0)	Graphics calculator (same brands)
Techniques available for solving problems in examinations	Pen-and-paper Technology-(Graphical, Numerical and Symbolic)	Pen-and-paper Technology-(Graphical and Numerical)

Table 1: Comparison of two cohorts of students

CAS students were familiar with a rubric designed to guide practice for writing solutions in a CAS classroom. The RIPA rubric (Ball and Stacey, 2003) was created in response to students' and teachers' needs. RIPA promoted use of mathematical notation rather than calculator syntax and the recording of reasons (R), information and inputs (I), a plan for the solution path (P) and some answers (A) in written records. If students include reasons, a plan and calculator inputs (using mathematical notation) then we expect more words in students' solutions. All teachers in the research project stressed the importance of clearly communicating written records.

SAMPLE EXAMINATION SCRIPTS AND QUESTIONS ANALYSED

The sample written records to be discussed are from the entire 78 Mathematical Methods (CAS) "examination 2" scripts and a stratified random sample of 78 Year 12 Mathematical Methods "examination 2" scripts. The random sample of non-CAS scripts was matched to the achievement of the CAS scripts, as the purpose of this paper was not to compare the relative achievements of the two groups.

Questions 3i and 3ii (see Figure 1), common to both the CAS and non-CAS examination, are discussed in this paper and compared to results from the initial analysis (Ball, submitted) of Question 1b. Students needed to provide reasoning to show two given results and find the coordinates of a point of intersection of two graphs.

VCE Mathematical Methods (CAS) Pilot Study Examination 2 (abbreviated questions)

Q 1:… According to Fitts' Law, for a fixed distance traveled by the mouse, the time taken, in seconds, is given by $a - b\log_e(x)$, $0 < x \leq 5$, where x cm is the button width and a and b are positive constants for a particular user…

Q1b. Mickey decides to find the values of a and b for his use. He finds that when x is 1, his time is 0.5 seconds and when x is 1.5, his time is 0.3 seconds. Find the exact values of a and b for Mickey.

Q 3: The diagram shows the curve whose equation is $y = \frac{1}{2}(2x^4 - x^3 - 5x^2 + 3x)$ and the normal to the curve at A where x=1.[Graph shown with intersections at A and B]

Q3i. Show that the equation of this normal is $y=x-1.5$.

Q3ii. Show that this normal is a tangent to the curve at point B and find the exact values of the coordinates of B.

Figure 1: Examination questions analysed (Q1b, Q3i, Q3ii) (VCAA, 2002)

Analysis of these two problems enabled investigation of whether observed differences in features of written records for Question 1b were also apparent in the written records for these additional two common questions. This provides some insight into whether or not features of CAS written records appear as part of a new practice that students have developed for recording mathematical thinking or just in response to particular problems.

Four features of written records

Following Ball (submitted), the use of words and mathematical notation in students' written records are categorised in two ways, as shown in Table 2. The length of written records was also measured, simply as the number of lines on the page that contained any working. It was also noted whether solutions used function notation (i.e. $f(x)$). As an example, the length of the written record in Figure 2 was seven. The line showing $y = \frac{1}{2}$ follows from the statement x=1 which shows that it is a separate line. The record is categorised as M′ as it contains evidence of non-standard notation. The student has used $|$ to indicate substitution and also given the CAS syntax $\text{solve}(y-(-\frac{1}{2}) = 1(x-1), y)$ in solving for *y*. The solution is classified as W because it contains words (importantly "solve"). It does not use function notation. More sophisticated measures and definitions did not seem to give different results to these simple ones and were harder to implement consistently.

Code	Written record
M	contains standard mathematical notation only
M′	contains some non-standard mathematical notation
W	contains one or more words that can be found in a dictionary
W′	does not contain any words that can be found in a dictionary

Table 2: Codes for categories of written records (Ball, submitted)

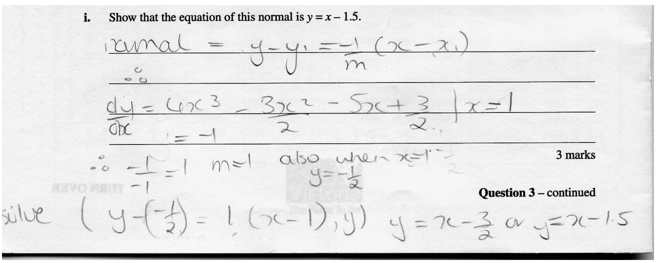

Figure 2: Example of written record containing non-standard notation

RESULTS AND DISCUSSION

Use of standard and non-standard mathematical notation

There were few instances of non-standard notation evident in the written records for the two problems (see Table 3). Q3i and Q3ii each had two occurrences of non-standard notation in CAS solutions and Q3ii had one instance of non-standard notation in a non-CAS solution when a student recorded the name of the calculator program (FCTPOLY2) used to factorise a polynomial. There is no statistical difference between CAS and non-CAS students (χ^2 corrected = 0.44, d.f. = 1, p = 0.506). Non-standard notation given in CAS written records was in the form of a CAS instruction for solving or for substitution, both of which are shown in Figure 2. This limited use of non-standard notation was also reported by Ball for Q1b.

Features of Solutions	Solutions to Q3i		Solutions to Q3ii	
	CAS	NonCAS	CAS	NonCAS
M+W	64	65	65	58
M+W'	9	12	6	15
M'+W	2	0	2	1
M'+W'	0	0	0	0
Written record contained:				
‘solve’	4	0	22	1
‘substitute’	16	7	20	7
‘simultaneous’	0	0	1	0
‘Define’	1	0	0	0
‘and’	4	5	8	5
‘equation’	17	21	26	10
function notation	13	10	15	9

Table 3 Number of solutions exhibiting specified features of written records

The results for the two questions discussed in this paper and previous analysis of Q1b suggest that most students are careful to use standard mathematical notation in recording their solutions even though they have learned mathematics with CAS and may have used CAS for various steps within solutions. In class, the non-standard notation we observed was generally of the nature of generic (not brand-specific) CAS syntax and as such is easily understood by someone with a mathematical background. We expect that some of these currently non-standard notations may become standard.

Length of written records

The lengths of written records for Q3i and Q3ii are shown in Figures 3 and 4. Q3i had a number of shorter CAS written records. The average length of written records for each question (n=78 for each) was found to be significantly greater for non-CAS students in Q3i (CAS 5.5, Non-CAS 8.3, t=6.4, d.f.= 154, p=0.000) but almost the same for the two groups of students in Q3ii (CAS 6.9, Non-CAS 7.0). For Q1b (n=78) the average length was found to be greater for non-CAS written records (CAS 5.2, Non-CAS 7.2). If we only consider written responses of students that responded to the task (i.e. not length 0), the same results still hold. These results reflect the fact that CAS students are able to perform intermediate routine steps with CAS. This is an important consideration for teachers and students because in these examinations students need to communicate enough appropriate working to access intermediate marks if their final answer is incorrect. The CAS students in this study were aware that examiners needed to be able to follow the working documented in written records. This sort of consideration will mould evolving practice.

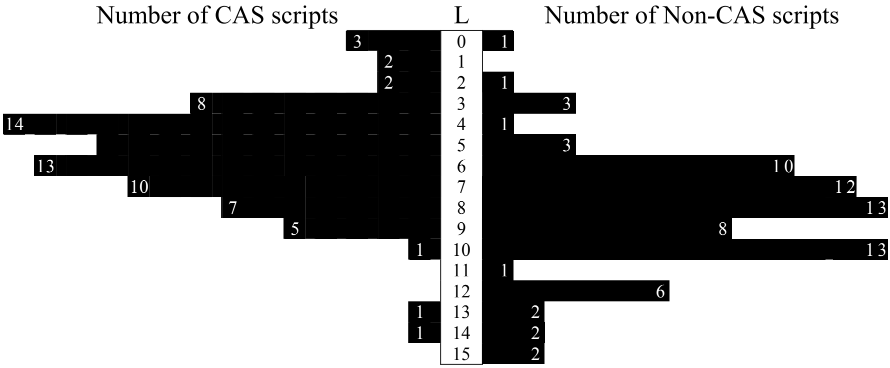

Figure 3 Lengths (L) of CAS and non-CAS written records for Q3i

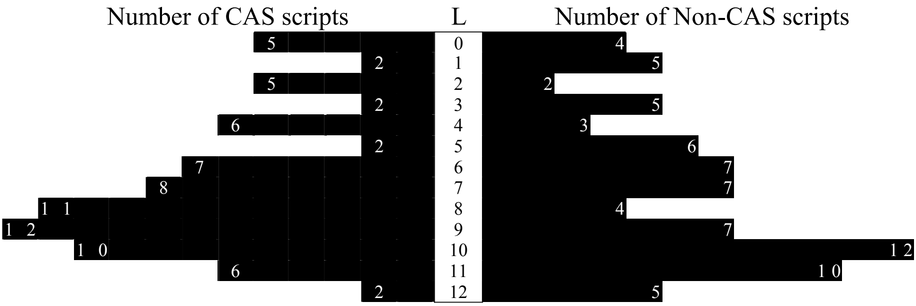

Figure 4 Lengths (L) of CAS and non-CAS written records for Q3ii

Use of words

For Q3i and Q3ii, the percentage of written records containing words was greater for students who learned mathematics with CAS than for non-CAS students. Combining the data in Table 3 for both questions shows 89% of CAS solutions and 82% of non-CAS solutions contained words, a notable overall difference (χ^2 corrected =3.10, d.f.=1, p=0.078). Overall, the high usage of words and in particular use of the word "solve" would suggest that access to CAS is impacting on the written records and possibly also the way in which students think about mathematical commands. The words used in CAS and non-CAS records had interesting differences. Non-CAS written records tended to contain the words “substitute”, “and”, or “equation” – this is familiar to us, but we had not noticed the restricted range of words until this study. Surprisingly, use of the word “solve” was mainly observed in CAS written records with nearly 30% of CAS written records for Q3ii including this word but few non-CAS records. The use of “solve” in CAS written records could be attributed to a number of factors. We have observed that students often write this word when they are recording CAS syntax. We propose that these students may be thinking about solving at a more global level than non-CAS students. Non-CAS students need to attend to the intermediate steps of a solution, rather than thinking about “solving” overall. For example, to solve a quadratic equation they may first consider rewriting it so that they have a quadratic expression equal to zero, then make an attempt to factorise the quadratic expression and so the focus would be on factorizing rather than thinking about the overall process which is solving. A CAS student can just recognise that they need to solve.

Use of function notation

CAS written records also contained more use of function notation than non-CAS written records for both questions (see Table 3), although the difference is not statistically significant (χ^2 corrected =1.95, d.f.=1, p=0.163). Given that these results occurred for all three questions, this could indicate a new practice for recording. It probably results from the ease of use of function notation with CAS, and the technical benefits of defining functions explicitly for subsequent solving, substituting etc. CAS students may get into the habit of recording a function using function notation initially in their written records to facilitate later use of the inbuilt “define” feature in the solution, and hence access to simplified CAS input.

CONCLUSION

The analysis of an additional two questions generally supports the observations of Ball (submitted) that CAS solutions will generally be shorter than non-CAS solutions, that they will contain more words and use function notation more. Some changes may occur in the mathematical notation that is regarded as acceptable, but there need be no fears that students will replace standard notation by incomprehensible machine-speak. There is evidence that a new mathematical practice is evolving with this tool. Teachers and others need to actively guide this evolution in desired directions.

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