A STATEMENT, THE CONTRAPOSITIVE AND THE INVERSE: INTUITION AND ARGUMENTATION

Samuele Antonini

Department of Mathematics - University of Pisa, Italy

The equivalence between a statement and its contrapositive is so obvious for an expert that, usually, he does not need any explanation. In this paper, we shall examine the argumentations which students produce in order to justify a statement that, in their opinion, is equivalent to a given statement. We shall observe that the most common argumentations come out from the effort to adjust the proof of the first statement to the second one. Analysing these argumentations, it will emerge that for the students the (false) equivalence between a statement and its inverse is intuitive and the (true) equivalence between a statement and its contrapositive is not intuitive.

INTRODUCTION

In this paper, we investigate the equivalence between a statement and its contrapositive from cognitive and didactical points of view. Given a statement $p \rightarrow q$, the contrapositive is the statement $\neg q \rightarrow \neg p$, the inverse is $\neg p \rightarrow \neg q$ and the converse is $q \rightarrow p$. A statement and the contrapositive are equivalent, then, if we have proved the statement, the contrapositive is proved too. A statement and the inverse are not equivalent; it happens that a statement is true but the inverse is false; in the same way a statement and the converse are not equivalent.

Usually, in school practice, teachers and text-books do not pay much attention to the equivalence between $p \rightarrow q$ e $\neg q \rightarrow \neg p$, probably because it is considered to be obvious: it is common opinion that this equivalence is a natural way of thinking that can be spontaneously used in mathematics. For example, in a discussion about proof from didactical point of view, an university teacher stated that "*the proof by contradiction is based on the fact that $p \rightarrow q$ and $\neg q \rightarrow \neg p$ say the same thing*". It was so obvious for him that he did not need any explanation. Moreover, the link between the two statements was so strong for him that he did not seem to be talking about a logic equivalence: the two statements "*say the same thing*", as they were two different forms of the same statement.

On the other side, some researches in education dealt with this equivalence and showed that it seems to be content dependent and, in some cases, to be not so obvious. For a rich bibliography, see Fischbein (1987, pp. 72-81).

Moreover, some researches in mathematics education have pointed out some epistemological, cognitive and didactical obstacles for students when they have to understand or to produce proofs by contradiction, that is proofs whose validity is based on the equivalence between a statement and the contrapositive. See, for example, Wu Yu et al. (2003), Antonini (2003a, 2003b, 2001), Bernardi (2002), Reid

(1998), Epp (1998), Thompson (1996), Barbin (1988), Leron (1985), Freudenthal (1973).

The aim of this work is to investigate the argumentations proposed by secondary school students (grade 10) when they justify the validity of the contrapositive, the converse and the inverse of a given statement. In the theoretical framework on intuitions exposed by Fischbein (1987, 1982), analysing the argumentations, we can examine and discuss the intuitiveness of these equivalences.

THEORETICAL FRAMEWORK

This study is based on the theoretical framework on intuitions exposed by Fischbein (1987, 1982). In this work, an intuition is "*a representation, an explanation or an interpretation directly accepted by us as something natural, self-evident, intrinsically meaningful, like a simple, given fact"* (Fischbein, 1982, p.10).

Moreover, it is very interesting the following classification:

> *"We have, then, three different kinds of convictions. One is the formal extrinsic type of conviction indirectly imposed by a formal (sometimes a purely symbolical) argumentation. The second is the empirical inductive form of conviction derived from a multitude of practical findings which support the respective conclusion. The third is the intuitive intrinsic type of conviction, directly imposed by the structure of the situation itself" (Fischbein, 1982, p.11).*

If one knows something intuitively, "*he will not feel the need to add something which could complete or clarify the notion (for instance an explanation, a definition, etc.)"* (Fischbein, 1982, p.10). For example, what the university teacher said about proof by contradiction (see above) is, for him, "*something natural, self-evident, intrinsically meaningful, like a simple, given fact"*, i.e. an intuition.

From the mathematical point of view, the proofs of the equivalence between a statement and the contrapositive are based on the principle of excluded middle and they can be formalized in the logic of proposition, for example through the truth tables. We underline that these are meta-theoretical proofs: the object of these proofs is the validity of a statement in relation to the validity of another statement.

Then, it will very interesting to observe the explanations that students propose to support or to reject the equivalence between a statement, the contrapositive or the inverse or the converse.

We expect the students not to produce meta-theoretical argumentations. We rather expect them to propose specific argumentations that depend on the given statement. Through the analysis of these argumentations we shall investigate the intuitiveness of the equivalence between a statement and the contrapositive and the non equivalence between a statement, the inverse and the converse.

METHODOLOGY

This work is a part of a wider study about proof by contradiction exposed in Antonini (2003a) and then is based on the methodology of that research. In particular, for the research explained here, we observed 46 secondary school students (grade 10) through discussions, reports written by the students about the discussions, a test, and a report in which students had to justify their answers in the test. In this paper, we describe and analyse what emerged from the discussions and from the reports about two particular mathematical problems. To make the exposition clear, we discuss the results according to the mathematical problem they refer to.

THE PROBLEM OF THE PARALLEL LINES

The episode is part of the regular didactical activity in two classes. The students have just proved the following statement (that we call the *main statement*):

Main statement: *Let r and s be two intersecting lines and let t be a transversal. Then the alternate interior angles are different.*

The proof proposed by the students is based on the theorem of the exterior angle: in a triangle an exterior angle is bigger than every non adjacent angle. In this case, in the triangle ABP, we can say that $\alpha > \beta$, and then $\alpha \neq \beta$.
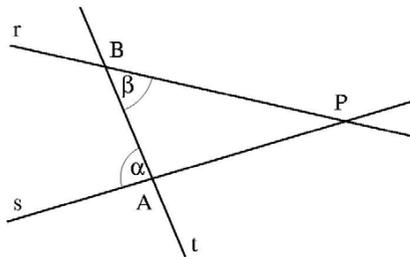
After the proof, the teacher asked the students: *can you formulate a statement which contains parallel lines in the hypothesis or in the thesis, and which does not need to be proved because we already know that it is true from the validity of the main statement?*

During the discussion, students proposed two statements, the inverse and the contrapositive:

inverse: *If r is parallel to s then the two alternate interior angles are equal*

contrapositive: *If the alternate interior angles are equal, then r is parallel to s.*

At home, the students wrote a report in which they exposed their opinions. It emerged that 20% of the students chose the contrapositive, 20% of the students chose the inverse, 50% of the students chose both the statements and 10% of the students completely misunderstood the task or the statement.

The most common argumentations to justify the statement come out from the effort to adjust the proof of the main statement to the inverse or to the contrapositive. The students, instead of producing meta-theoretical argumentations, proposed argumentations based on the same theoretical reference (that is the exterior angle

theorem) of the proof of the main statement. By this kind of argumentation students could support the validity of both the inverse and the contrapositive: these argumentations do not allow to identify the statement equivalent to the given one. For example, Fabiana, after the proof of the main statement, writes:

"In this way, using the exterior angle theorem, we can prove and be sure that two intersecting lines and a transversal make a triangle and the two alternate interior angles are different.

Probably, to prove that two parallel lines and a transversal [make] two equal alternate interior angles, we have to think in the same way, using the ideas and the theorems previously proved."

Fabiana thinks that, to prove the inverse, "*we have to think in the same way*". Then, she explains what she means in the following way:

"We can prove the exterior angle theorem, that is that $\alpha > \beta$ and then that $\alpha \neq \beta$ just when we have a triangle formed by the intersecting lines. Then, when the lines are parallel, they cannot make a triangle because they do not intersect each other and then we cannot apply the theorem and then we cannot prove that $\alpha > \beta$ and then, if the two lines are parallel, $\alpha = \beta$."

In the same way, Fabiana supports the contrapositive using the exterior angle theorem:

"If I have in the hypothesis that $\alpha = \beta$ and in the thesis that r//s I can reverse the argument because I know that $\alpha = \beta$ and then α is not bigger than β then I can deduce that the lines do not make a triangle and then I can not apply the exterior angle theorem."

Explanations like this are very common. We can identify two elements in these argumentations: one about the content, one about the structure. Firstly, they are based on the same argument of the proof of the main statement, that is the exterior angle theorem. Secondly, the structure of these argumentations is always the following: *if we can apply a particular theorem, we deduce a certain conclusion, otherwise we deduce the opposite of that conclusion*.
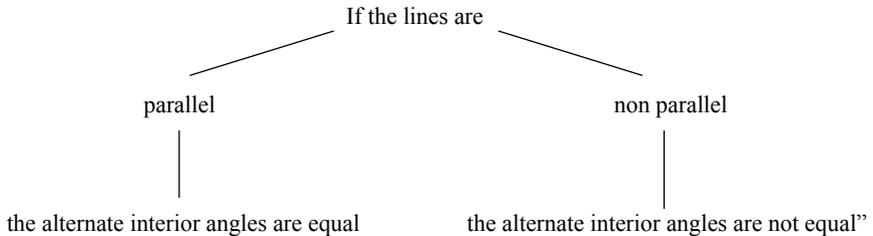
Other students answered in a very different way: they proposed the inverse of the main statement and they did not need to give any argumentation for this choice. The following episodes are just two examples of many similar protocols:

158.   Achille: if [the lines] are parallel, the angles are equal.

159.   Teacher: You are thinking about the first statement [the inverse]. How are you going to prove it?

160.   Achille: because… if they intersect each other … is so… if they do not intersect … it is not so…

161.   Teacher: Is this a proof?

162.   Achille: No.

163.   Teacher: But it is a convincing argument for you.

164.   Achille: Eh… if we begin from two intersecting lines… the angles are different… if we begin from non intersecting lines, the angles will be equal!

Achille is convinced, it is so evident for him that he does not need to propose any further argumentation. Also Gianna, in the written report, answers in a similar way:

"In my opinion, it is like a schema:

If the lines are

parallel   non parallel

the alternate interior angles are equal   the alternate interior angles are not equal"

According to Fischbein (1987, pp. 72-81), we can conclude that for Achille and Gianna the equivalence between a statement and the inverse is an intuition. In this regard, the Franca protocol is very enlightening. She proposes a logic argumentation based on the truth tables to justify the equivalence between a statement and the contrapositive[1]: her explanation is the only one we can consider a proof. Nevertheless, she also asserts the equivalence between the statement and the inverse:

"But, in my opinion, it is correct to say also that, if r//s then we do not have any elements to prove that $\alpha \neq \beta$ and then we have to accept the thesis $\alpha = \beta$."

Even if Franca writes a meta-theoretical proof to support the equivalence between a statement and the contrapositive, she proposes an argumentation based on an intuition which can not be formalized in logic: "*we do not have any elements to prove that $\alpha \neq \beta$ and then we have to accept the thesis $\alpha = \beta$*". We explain these phenomena describing two *convictions* (Fischbein, 1982) that can coexist: the first one is the intuitive *conviction* that a statement and the inverse are equivalent; the second one is the non intuitive *conviction* that a statement and the contrapositive are equivalent. Other students, like Franca, write that the contrapositive is the equivalent statement, but the more intuitive *conviction* emerges and they affirm the equivalence of the inverse too. With words of Fischbein, the first one is an *intuitive intrinsic type of conviction*, the second one is a *formal extrinsic type of conviction*. We remark that the last one does not seem to have any effects on the first *conviction* which can still continue to be an obstacle.

---

[1] We remark that the use of the truth tables was never used for problems of this kind neither by these students nor by the teacher.

THE PROBLEM OF THE QUADRILATERAL

After the episode of the parallel lines, the teacher spent some lessons to justify the equivalence between a statement and the contrapositive and the non equivalence between a statement and the inverse. The teacher's explanations were based on two different contents: the proof by the truth tables and the familiar examples expressed in natural language. Three weeks later, students were asked to answer a test with two questions:

1) Write down the hypothesis and the thesis of the statement: *If the angles of a quadrilateral are equals, the diagonals are equal.*

2) Let us suppose that the statement of question 1) is proved. Which of the following statements is consequently proved?

     a) If the angles of a quadrilateral are not equal, the diagonals are different.

     b) If the diagonals of a quadrilateral are different, the angles are not equal.

     c) If the diagonal of a quadrilateral are equal, the angles are equal.

Question 1) is useful for us to understand if the answers to question 2) could be dependent on an incorrect identification of the hypothesis or of the thesis. Every one of the 43 students answered correctly to the first question, then we have to look for other elements to explain the students' difficulties with the second question. Calling "*main statement*" the one about the quadrilateral in question 1), the statement in a) is the inverse, the statement in b) is the contrapositive and the statement in c) is the converse. The data of the answers to the second question are in the table:

inverse	contrap.	converse	inverse and contrap.	inverse and converse	contrap. and converse	Inverse, contrap. and converse
15	9	8	6	1	2	2
35%	21%	19%	14%		11%	

We underline that, in this case, the inverse is false; in the problem of the parallel lines, the inverse is true even if it can not be proved from the main statement. Nevertheless, a lot of students chose the inverse as their answer, both in the problem of the parallel lines and in the problem of the quadrilateral.

After the test, the teacher asked the students to write down a report explaining the reasons for their answers.

In this case, the students did not have the proof of the main statement, then they could not use this proof to justify their choices like they did in the problem of the parallel lines. Then they produced a lot of examples, in order to recognize and to argument

---

the correct statement. We remark that, in this way, the students looked for the valid statement among the three proposed: they did not refer to the equivalence between the chosen statement and the main one. Just one student used correctly the truth tables to answer the question. Nevertheless, Marilena, after having proved that the main statement was not equivalent to the inverse using the truth tables, wrote that "*we can not deduce [the inverse] from the proved theorem, even if it is intuitively true*". The presence of the intuitive *conviction* about the inverse is evident but now Marilena checks her intuition through a formal tool. However, the fact that a statement and the inverse are not equivalent is a *formal extrinsic type of conviction*, but it is not yet an intuition.

CONCLUSIONS

We have analysed the argumentations that students proposed to justify the equivalence between a statement and the contrapositive or the inverse: it emerged that most of these argumentations come from the proof of the main statement. When the students did not know this proof they have justified such equivalence producing a set of examples. In every case, their aim was to look for a valid statement and not a statement equivalent to the given one.

According to Fischbein (1987, pp. 72-81), we can affirm that the (false) equivalence between a statement and the inverse is an intuition while the (true) equivalence between a statement and the contrapositive is not an intuition. Moreover, the argumentations proposed by the teacher, based on the truth tables and on familiar examples, did not help the students: just few students used these type of explanations to justify their answers, and even for them the topic did not become intuitive.

According to Antonini (2003a, 2003b), Thompson (1996) and Freudenthal (1973), it is fruitful to set up situations in which students spontaneously produce *indirect argumentations*, that is argumentations like "*...if it were not so, it would happen that...*" (Antonini, 2003b). Moreover, it is important to guide the students to the awareness of the structure of their argumentations, so that the knowledge of the equivalence between a statement and the contrapositive and of the non equivalence between the statement and the inverse becomes an intuitive knowledge. As Fischbein wrote:

> "*The training of logical capacities is a basic condition for success in mathematics and science education. We refer not only to a formal-algorithmic training. The main concern has to be the conversion of these mental schemas into intuitive efficient tools, that is to say in mechanisms organically incorporated in the mental-behavioral abilities of the individual*" (Fischbein, 1987, p.81).

It is important that the contrapositive becomes a way of thinking, an intuitive knowledge, in order to assume the features of a thinking tool.

References:

Antonini,S.(2001), Negation in mathematics: obstacles emerging from an exploratory study, *Proceedings of the 25th PME Conference, Utrecht, The Netherlands*, vol. 2 pp. 49-56.

Antonini,S.(2003a), *Dimostrare per assurdo: analisi cognitiva in una prospettiva didattica*, Tesi di Dottorato, Dipartimento di Matematica, Università di Pisa.

Antonini,S.(2003b), Non-examples and proof by contradiction, *Proceedings of the 2003 Joint Meeting of PME and PMENA, Honolulu, Hawai'i, U.S.A.,* vol. 2 pp. 49-55.

Barbin,E.(1988), La démonstration mathématique: significations épistémologiques et questions didactiques, *Bulletin APMEP* 366, pp. 591-620.

Bernardi,C.(2002), Ricerche in Didattica della Matematica e in Matematiche Elementari, *Bollettino Unione Matematica Italiana*, Serie VIII, Vol. V-A pp. 193-213.

Epp,S.S.(1998), A Unified Framework for Proof and Disproof, *The Mathematics Teacher,* v. 91(8) pp. 708-713.

Fischbein,E.(1982), Intuition and proof, *For the Learning of mathematics* 3 (2), Nov., pp. 8-24.

Fischbein,E.(1987), *Intuition in science and mathematics*, Dordrecht: Kluwer.

Freudenthal,H.(1973), *Mathematics as an educational task*, Reidel Publishing Company: Dordrecht, Holland.

Leron,U.(1985), A Direct approach to indirect proofs, *Educational Studies in Mathematics* vol. 16 (3) pp. 321-325.

Reid,D.&Dobbin,J.(1998), Why is proof by contradiction difficult?, *Proceedings of the 22th PME Conference, Stellenbosch, South Africa* vol. 4 pp. 41-48.

Thompson,D.R.(1996), Learning and Teaching Indirect Proof, *The Mathematics Teacher* v. 89(6) pp. 474-82.

Wu Yu,J.,Lin,F.,Lee,Y.(2003), Students' understanding of proof by contradiction, *Proceedings of the 2003 Joint Meeting of PME and PMENA, Honolulu, Hawai'i, U.S.A.,* vol. 4, pp. 443-449.