SECONDARY MATHEMATICS TEACHERS' KNOWLEDGE CONCERNING THE CONCEPT OF LIMIT AND CONTINUITY

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The present study aims to explore the secondary teachers' understanding and reasoning about the concepts of limit and continuity. The findings suggest that teachers have not developed a rich relational understanding of these notions. They exhibited disturbing gaps in their conceptualizations of limit and continuity.

INTRODUCTION

The concepts of limit and continuity are of fundamental importance in the learning of Mathematical Analysis and has been the focus of attention for many researchers on mathematics education. Students do not appear to understand these notions easily. They face cognitive difficulties because of the richness and complexity of them. There are a lot of studies dealing with these difficulties and didactical approaches of these concepts (e.g. Artigue, 1997; Cornu, 1981, 1991; Ferrini-Mundy & Graham, 1991; Mamona-Downs, 2001; Schwarzenberger & Tall, 1978; Sierpinska, 1985; Vinner 1991). A common conclusion of these studies is that the majority of the students have deficient understanding of these concepts, even at a more advanced stage of their studies. However, not enough attention has been drawn to the secondary teachers' knowledge of these notions. Since a course of pre-Calculus is contained in the curricula of school mathematics, it is important to explore the teachers' understanding and reasoning of the concepts of limit and continuity. This is the goal of the present study, which is a part of a larger research on the extent and sufficiency of the subject matter knowledge of secondary mathematics teachers.

THEORETICAL BACKGROUND

There are many studies on the subject matter knowledge of the teachers (Ball, Lubienski & Mewborn, 2001, p.448). The majority of them concerns with the preservice and the elementary teachers. It would appear that very little is known about the extent or the sufficiency of the subject matter knowledge of secondary mathematics teachers. Perhaps the cognitive competence of the secondary mathematics teachers is taken for granted, since they teach the object which they have studied during their undergraduate studies. However research results suggest that this is not always true (e.g. Ball, 1990; 1991; Norman, 1992).

Multiple frameworks exist for thinking about mathematics understanding. Skemp (1976, 1978) distinguished the knowledge in mathematics in instrumental and relational knowledge. The instrumental understanding refers to an algorithmic understanding of a concept or process and the relational understanding comes from an understanding of deeper relationships among the concepts and processes associated with a particular concept or situation. After Skemp's work other classifications of mathematical understanding follow. Hiebert (1986) distinguished in procedural and conceptual mathematical understanding, Ball (1988) in knowledge *of* and *about* mathematics and others. Shulman (1986) identified two components of the professional knowledge of teachers: content knowledge and pedagogical content knowledge. For Shulman, content knowledge

> refers to the amount and organization of knowledge per se in the mind of the teacher… To think properly about content knowledge requires going beyond knowledge of the facts or concepts of a domain. It requires understanding the structures of the subject matter
>
> (*ibid,* p. 9)

In pedagogical content knowledge Shulman includes "the most useful forms of representation of those ideas, the most powerful analogies, illustrations, examples, explanations, and demonstrations – in a word, the ways of representing and formulating the subject that make it comprehensible to others" (*ibid*, p. 9).
Shulman (1987, pp. 17-18) emphasizes that lack of content knowledge was the underlying reason for less effective teaching. Other studies have indicated that teachers' cognition drives their instructional practice (Carpenter, 1989; Fennema, Carpenter, & Peterson, 1989).

In this study we focus on the extent and the sufficiency of the subject matter knowledge of the secondary teachers on the concepts of limit and continuity.
Particularly, we attempt to trace their actual content knowledge and their pedagogical content knowledge as far as the two above mentioned concepts are concerned.

METHOD

The study is based on 15 secondary mathematics teachers, each one working towards a master degree in mathematics education. All of them had an undergraduate degree in Mathematics. During their undergraduate studies they attended courses about Calculus, based on books like M. Spivak (1967). They all had experience in teaching Mathematical Analysis in secondary school. During their studies for the master degree they attended, among others, a 12 weeks course on the teaching of Calculus. During this course, the concepts and difficulties attached to limit and continuity were extensively discussed. At the end of this course the teachers answered a questionnaire and afterwards we had an interview with each one of them and discussed his/her answers.

FINDINGS

The teachers were asked the following questions:

i) Give the definition of limit of a sequence. How would you describe it to a student so that he would understand it?

ii) Write two student's misconceptions and explain how you would deal with them.

iii) Let a sequence $(a_n)_{n\in\mathbb{N}}$. Check if the next two statements are equivalent with $\lim_{n\to\infty} a_n = 0$.

P1: The sequence $(a_n)_{n\in\mathbb{N}}$ has terms with absolute value as small as possible.

P2: There exists a natural number n_0 such that for every $\varepsilon > 0$ and for every $n \geq n_0$ we have $|\alpha_n| < \varepsilon$.

iv) Let $A \subseteq \mathbb{R}$, and $f : A \to \mathbb{R}$. Give the definitions of continuity of f at a point $x_0 \in A$ and on A. How would you explain these concepts?

v) Write one student's misconception about the continuity and explain how you would deal with it.

Judging by the teachers' answers in the questionnaire and by their interviews, several problems came out concerning the content knowledge and the pedagogical content knowledge of the concepts of limit and continuity. According to the relevant literature, many of these problems had also appeared in the case of students (Cornu, 1991; Sierpinska, 1985; Vinner, 1991 and others). Some of the main problems were:

a) Difficulty in understanding the meaning of an inequality in the frame of complex statements such as related to the definition of limit and continuity.
A teacher, answering the question (ii), writes:

> Another question that shows misconception on behalf of the students is: "It is a mistake to put $|\alpha_n - \alpha| \leq \varepsilon$ in the definition of the convergence of a sequence?"
> My answer is that of course it is a mistake, since it means that there is the case $\alpha_n = \alpha$.

Another teacher answering the question (v) writes:

> I draw their attention [about the definition of continuity] to the fact that $|f(x) - f(x_0)| \leq \varepsilon$ and not $|f(x) - f(x_0)| < \varepsilon$, as the definition of the limit, since it can be $f(x) = f(x_0)$.

From the above answers we see that these teachers have difficulty to understand the meaning of these inequalities in the frame of the definitions of a limit and continuity. They don't understand that the statements $\forall\ \varepsilon > 0\ \exists\ n_0 \in \mathbb{N} : \forall\ n \geq n_0\ |\alpha_n - \alpha| < \varepsilon$ and $\forall\ \varepsilon > 0\ \exists\ n_0 \in \mathbb{N} : \forall\ n \geq n_0\ |\alpha_n - \alpha| \leq \varepsilon$ mean exactly the same and of course the difference between the inequalities $|f(x) - f(x_0)| \leq \varepsilon$ and $|f(x) - f(x_0)| < \varepsilon$ is not that in the first one it can be $f(x) = f(x_0)$.

b) From answers, like the ones mentioned above, it came out also that some of them believe that a convergence sequence doesn't reach its limit.

c) Difficulties had appeared in the correct understanding of the meaning of the quantifiers. Almost all of the teachers answered in the question (iii) that the statement P2 is equivalent with $\lim_{n\to\infty} a_n = 0$. From the interviews followed that they believed that the change of the order of the quantifiers in the statements

$\forall\ \varepsilon > 0\ \exists\ n_0 \in \mathbb{N} : \forall\ n \geq n_0\ |\alpha_n - \alpha| < \varepsilon$ and $\exists\ n_0 \in \mathbb{N} : \forall\ \varepsilon > 0\ \forall\ n \geq n_0\ |\alpha_n - \alpha| < \varepsilon$
does not change the meaning. For them the above statements are identical.
Many of them also believed, as we observed from the answers to the question (i) and from the interviews that it is a mistake if someone doesn't notice that n_0 depends on ε. For them the order of the quantifiers isn't enough.

d) From the answers to the question (i), we observed that a great number of them have not understood right the relation between ε and n_0 in the definition of the convergence of a sequence. They believe that this relation is a 1-1 function and some of them note that this function is decreasing.
One of them writes:

> It has to be clear that n_0 depends on ε and for different ε there exists different n_0.

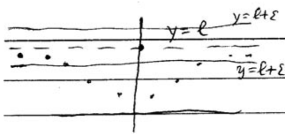
Another one notes:

> as ε decreases then n_0 increases.

e) From several answers it follows that some of them haven't understood sufficiently the role of the variables in the definition of the continuity.
Someone writes:

> We observe that for every x close to x_0 we can find an interval of $f(x_0)$ such that $f(x)$ to belong in this interval (for a suitable choice of ε)

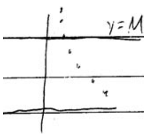
f) Some of the teachers, wanting to give graphically the convergence of a sequence, draw wrong graphs, like the following:



g) Difficulties were found out in reading a graph and in giving graphically a statement.
A teacher draws the graph of the function $f(x)=\frac{1}{x}$, he explains how we take the graph of the sequence $a_n=\frac{1}{n}$, n=1, 2,… from it and he continues:

> … as n is increasing, the points of $a_n=\frac{1}{n}$ approach the line $y = 0$.

Another one draws the following graph



and he writes:

> I would ask [from the students] to find if there are points of the sequence over the line y = M for M > 0 as big as I please. My aim is to prepare the students to see the formal definition, $\lim_{n\to\infty} a_n = +\infty \Leftrightarrow \forall\, M > 0\ \exists\, n_0 \in \mathbb{N}$ such that $a_n > M\ \forall\, n \geq n_0$.

h) For the majority of these teachers the definition of the continuity of a function $f: A \to \mathbb{N}$ at a point $x_0 \in A$ is $\lim_{x \to x_0} f(x) = f(x_0)$, without concern if x_0 is an accumulation point of A or not.

i) Another problem which appears is that they don't have complete concept images for some notions. The concept image of the majority for a continues function is: *It is an interrupted curve;* without taking into consideration if the domain of the function is an interval or not.
Also in their explanations with graphs, they usually use simple pictures of monotonic functions.

j) Most of these teachers have difficulty in giving verbally, in a correct way, the formal definitions of limit and continuity.
For example, someone, trying to give verbally the definition of a limit of a sequence, writes:

> There exists a term of the sequence after which the difference of the sequence from a constant number became as small as we please.

Another one, giving verbally the concept of the continuity, writes:

> As x approaches to x_0 as close as we please, then $f(x)$ approaches to $f(x_0)$. Namely when the interval of x_0 becomes closer the same it will happen with the correspond interval of $f(x_0)$.

k) From the interviews we found out that the majority of the teachers have serious difficulties in giving symbolically the statements P1 and P2.

CONCLUSIONS

We noted before that there is a considerable body of literature addressing issues on students' understanding of the concepts of limit and continuity. These studies have indicated that they have a weak conception of these notions and usually exhibit purely instrumental understandings. Of course we would expect the secondary mathematics teachers to develop a reasonably rich relational understanding of them. While this study was limited in scope, several observations should be made about the teachers who participated in the study. We note that these teachers, in comparison with their colleagues, would be considered among the more mathematically experienced. Nevertheless, we observed that the majority of the teachers have not developed a rich relational understanding of the notions of limit and continuity. They exhibited disturbing gaps in their conceptualizations of these concepts. Their content knowledge was incomplete and it affected the pedagogical content knowledge.
We summarize below some of the more striking observations:
1. Most of them have difficulties in understanding multiquantified statements or fail to comprehend the modification of such statements brought about by changes in the order of the quantifiers.

2. Since their school-teaching is mostly based on specific cases or expressions of general concept, they tend to believed that all expressions of these general notions are similar to the ones they teach.
3. Some of them cannot read correctly a graph of function and give graphically a symbolic statement.
4. They cannot “translate” correctly from verbal to symbolic and vice versa.
5. They don’t always have complete concept images.

The results of this study suggest that the secondary mathematics teachers might not be fully be masters of their mathematical domain. Especially when it comes to understanding the notions of a limit and continuity and articulating their knowledge of them.

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