HELPING CHILDREN TO MODEL PROPORTIONALLY IN GROUP ARGUMENTATION: OVERCOMING THE 'CONSTANT SUM' ERROR

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We examine eight cases of argumentation in relation to a proportional reasoning task - the 'Paint' task - in which the 'constant sum' strategy was a significant factor. Our analysis of argument follows Toulmin's (1958) approach and in the discourse we trace factors which seem to facilitate changes in argument. We find that the arguments of 'constant sum strategists' develop in the presence of (i) a small group of children with conflicting strategies, (ii) a teacher-researcher who draws attention to the context of the problem as a resource for backing warrants, and (iii) a pictorial tool which can facilitate informal communication.

INTRODUCTION

It has long been 'known' that children's strategies and errors can be used as a starting point for effective mathematics teaching. Bell, Swan, Onslow, Pratt and Purdy (1985) suggest that 'conceptual diagnostic tests' can help teachers become aware of their pupils' strategies. Bell et al. (1985) suggest that after significant errors are identified, they can be resolved through 'conflict discussion'. Williams and Ryan (2001) also complemented research on diagnostic tools with research into arguments in discussion of small groups of pupils.

Accordingly, in this study, we complemented our previous research on diagnostic assessment of children's proportional reasoning (Misailidou and Williams, 2003) with analyses of their argumentation while working on selected ratio tasks. A 'small group collective argumentation' approach was designed and the data were analysed drawing primarily on discourse analysis and sociocultural theories of learning (Gee, 1996). Within this framework the significance of dialogic argument in supporting the group's knowledge-building was recognized (Yackel, Cobb and Wood, 1991 and particularly for proportional reasoning, Pesci, 1998) and also the importance of tools (Hershkowitz and Schwarz, 1999) and the critical role of the teacher in coordinating all the above (Williams and Ryan, 2001).

In this paper we report results on the development of arguments from eight groups working collaboratively on an item called 'Paint'. This item comes from a diagnostic test that we have constructed (Misailidou and Williams, 2003) and it had been found to produce a large number of interesting errors, including the rarely reported 'constant sum' error. A pictorial representation of the item was used as a tool for facilitating discussion because previous analysis had shown it to make an unusually significant impact on the item difficulty (Misailidou and Williams, 2003b).

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METHODOLOGY

The approach taken is multiple case study methodology (Yin, 2003) of discourses, in which each discourse is seen as an opportunity for each individual to develop arguments. We analyse each individual's discourse as a sequence of more or less complete arguments, that is, a series of statements relating data, conclusions, warrants and possibly backings: we describe this as their 'discursive path' (see next section for the definition of these terms).

We formed groups of pupils, whose responses to the 'Paint' item (see below) had been different on our previously administered and analysed diagnostic test, thus engineering conflict. Each group consisted of three pupils and was involved in a researcher-guided discussion. The children were set the task of persuading each other by clear explanation and reasonable argument of their answer. The researcher, adopting the teacher's role, established rules for the children's argument in order to facilitate participation in discussion. Moreover, she tried to ensure that the arguments for the errors were clearly voiced and justified. Finally, the children were asked to summarise what they had learnt.

The 'Paint' item was adopted from Tourniaire (1986):

Sue and Jenny want to paint together.

They want to use each exactly the same colour.

Sue uses 3 cans of yellow paint and 6 cans of red paint.

Jenny uses 7 cans of yellow paint.

How much red paint does Jenny need?

A pictorial representation of the item (shown in Figure 1 together with drawings from the pupils and the researcher) was used as a tool for facilitating discussion.

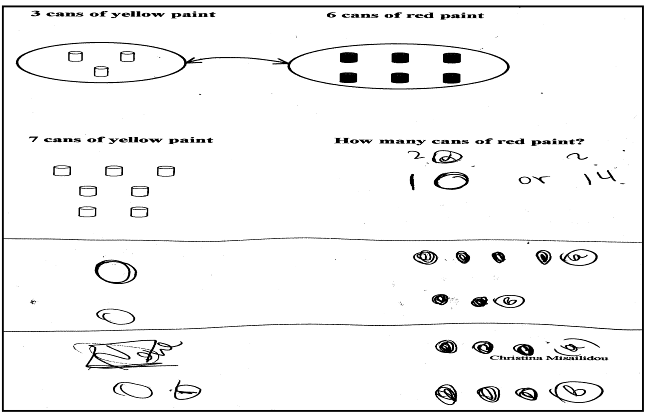

Figure 1: The 'pictures-sheet' used in the group discussions

The ‘Paint’ item provoked the highest frequency of the ‘constant sum’ (Mellar, 1987) error which is the incorrect answer ‘2’. A pupil applying the ‘constant sum’ strategy thinks that the sum of Sue’s cans should be equal to the sum of Jenny’s cans: 3+6=9 therefore 7+2=9. Pesci (1998) also reported the occurrence of this error in ‘colour mixing’ problems but generally it has not been given much attention in the literature. We believe that this error is important because of its very high frequency in certain tasks and because of the fact that a lot of pupils provide ‘adequate’ explanations for a ‘constant sum’ answer (i.e. they justify it by recourse to the context of the problem). Thus in this paper we will focus on the pupils that made this error in test conditions and then participated in the group discussions on ‘Paint’.

DATA ANALYSIS

The pupils’ arguments in each group discussion were recorded and analysed using Toulmin’s (1958) method. The utterances that made up the arguments were classified as ‘data’, ‘conclusions’, ‘warrants’ and ‘backings’. Data are the facts that are requested as a foundation for the conclusion. Warrants are the utterances which demonstrate that ‘taking the data as a starting point, the step to the…conclusion is an appropriate and legitimate one’ (Toulmin 1958, p. 98). Finally, backings are defined as the assurances that strengthen the authority of the warrants. The children's arguments were schematically represented using Toulmin's categories and then coded in order to assess changes in the provided arguments for adopted strategies.

Cobb’s distinction (2002) of children’s discourse about mathematics as ‘calculational’ and ‘conceptual’ was applied and refined to the context of proportional reasoning. Four categories were found relevant to classify pupils’ explanations in order to examine the development of their reasoning:

1. Numerical explanation: The pupil explains a numerical performance for finding an answer without justifying it contextually.

2. ‘Adequate’ explanation but non-multiplicative conceptualisation of the task: The explanation is defined as ‘adequate’ when the pupil connects numerical performances to contextual data. The pupil’s conceptualisation of the task context is defined as ‘non-multiplicative’ when it prohibits the construction of proportional relations.

3. ‘Adequate’ explanation and pre-multiplicative conceptualisation: The pupils have conceptualised the task ‘pre-multiplicatively’ when they can think relationally about quantities (for example more red paint than yellow is needed for a dark shade of orange), but not yet proportionally. (This is a slight variation of Resnick and Singer’s (1993) ‘protoquantitative relational reasoning’)

4. ‘Adequate’ explanation as a result of a multiplicative conceptualisation.

Pupils are not considered to *reason* proportionally if they just give an answer which is the result of a multiplicative calculational process, but rather when, additionally, their argumentation indicates multiplicative backing: i.e. we require evidence in the actual discourse and do not infer reasoning which is not stated. Hence, Category 4 is considered to be an indication of multiplicative reasoning. Category 3 is considered equally important since it is hypothesized that it can lead to Category 4 with appropriate teaching interventions. (Resnick and Singer, 1993)

'Discursive paths' for each of the pupils that took part in the group discussions were composed and studied. We define 'discursive path' as the evolution of the pupil's argumentation in the discussion. By combining discursive paths across group discussions we generalise patterns of changing arguments that we think of as 'changes of mind' (really they are changes of talk).

RESULTS

We present results from the eight groups that discussed the item 'Paint'. An example of a typical group discussion is given first and then the findings from all the groups are summarised. Our focus is the discursive paths of the group members that provided the answer '2' to the item in previous testing.

Example of a typical group discussion

Jane, Ann and Arpita (11 years old) were selected from the same class to form a discussion group because in previous testing they had provided three different responses to the 'Paint' item: '2' ('constant sum' strategy), '10' ('additive' strategy: 3+3=6 so 7+3=10) and '14' ('multiplicative' strategy: 2x3=6 so 2x7=14). The discussion, with the guidance of one of the authors, lasted 30 minutes and was audio taped. We will focus on Jane's discursive path through the discussion.

Initially, all the pupils recalled their response to the test and were invited to present an argument for it. Jane explained her answer '2'as: '6 add 3 is 9. That's how many paint cans Sue's got. And 7 add 2 equals 9. So, they should have exactly the same amount of paint. So, Jenny needs 2 cans of red paint.'

Her argumentation is represented using Toulmin's terminology in Figure 2. It is also labelled according to the categories mentioned above.

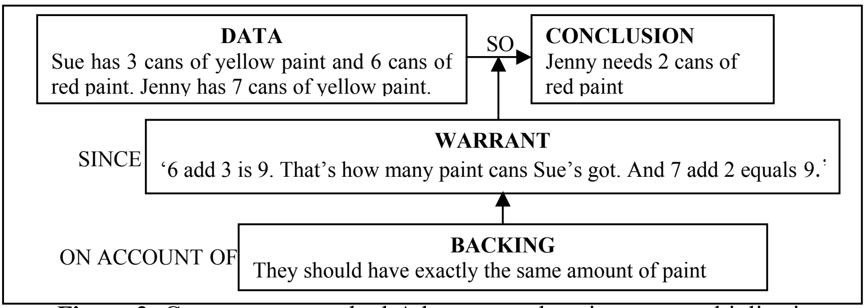

Figure 2: Constant sum method-Adequate explanation, non-multiplicative

After each pupil had presented their method, the researcher asked for reflection on all the methods: Jane insisted on her 'constant sum' method and influenced Arpita to adopt it. Then, in order to sustain and enrich the discussion, the researcher distributed the ‘pictures-sheet’ that is shown in Figure 1 as a tool that would facilitate discussion. From then on, she demonstrated throughout the discussion the use of the ‘pictures-sheet’: verbally (she referred, whenever possible, to the ‘pictures-sheet’) with her gestures (she pointed to it) and with her actions (she drew on it).

With the help of the ‘pictures-sheet’ a part of the discussion that was coded as ‘exploration of the context’ was initiated. This part was completed in three steps:

Step 1: The group realized that there is an essential condition that affects the answer to the problem. After the researcher’s prompt, Ann discovered the significant sentence: ‘They want to use exactly the same colour’.

Step 2: Prompted by the researcher, the group searched the meaning of the above condition: They established the difference between the ‘same colour’ and the ‘same amount of cans’.

Step 3A: The pupils focused on essential contextual elements such as the resulting shades by using different answers to the problem. At this point Ann realized that the answer ‘2’ was not reasonable:

> ‘It’s not exactly the same. *[She points with her pencil on the ‘pictures-sheet’ at Sue’s cans]* Sue has more red paint on the first one *[points at Sue’s red cans on the ‘pictures-sheet’]*…and then there’s more yellow on that one *[she points with her pencil on the ‘pictures-sheet’ on Jenny’s cans]*.

Step3B: The pupils clarified that the resulting (from the mixing of paints) colour was orange.

Following the ‘exploration of the context’ Jane changed her mind and declared that ‘2’ was not an appropriate answer. Her argumentation is presented in Figure 3.

DATA
Sue has 3 cans of yellow paint and 6 cans of red paint. Jenny has 7 cans of yellow paint.

SO

CONCLUSION
Jenny should not use 2 cans of red paint.

SINCE

WARRANT
[If Jenny uses 2 cans of red paint] ‘It would be darker there. *[pointing at her pictures sheet]* Sue’s would be darker.’

ON ACCOUNT OF

BACKING
She’s got more red paint *[than yellow. Jane points at Sue’s red cans and then at Sue’s yellow cans on the ‘pictures-sheet’]*.

Figure 3: Rejection of ‘constant sum’-Adequate explanation, pre-multiplicative

It is hypothesized that the essential elements that made Jane change her mind were:

1. The introduction of the ‘pictures-sheet’ and its use by the group and the researcher,
2. The ‘exploration of the context’ part of the discussion and

3. Ann’s argument about rejecting the answer ‘2’

After all of them had rejected the answer ‘2’, they seemed confused whether to accept ‘10’ or ‘14’. So the researcher introduced a new ‘extreme case’ to the problem by drawing on the ‘pictures-sheet’:

> ‘Let me try…another person, Maria. *[Draws 3 cans of red paint on her ‘pictures-sheet’]*. She has 3 cans of red paint. OK? With this method *[the multiplicative method]*…how much yellow paint does Maria need?’

This is an extreme case because if the additive (instead of the multiplicative) method is used, the resulting answer is ‘0’, so the resulting colour is red instead of orange. Thus the additive method does not work for this case.

After the group had found the answers for both methods and drawn them on the ‘pictures-sheet’ it seemed natural for them to reject the additive method and adopt the multiplicative one after offering adequate explanations for their choice. Jane’s argumentation for finally accepting the answer ‘14’ (the ‘third stage’ of her discursive path) is not of interest for this paper. We just provide her backing, which indicates a multiplicative conceptualisation of the task at the end of the discussion: ‘With doubling…we will make exactly the same shade of orange [for every case].’

In summary, Jane’s participation in the discussion group seem to have facilitated her way towards proportional reasoning as indicated by her ‘gradually multiplicative’ argumentation: ‘Constant sum’ method and non-multiplicative conceptualisation → Rejection of ‘constant sum’ method and pre-multiplicative conceptualisation → Multiplicative method and multiplicative conceptualisation.

It is hypothesized that the critical factors that lead to her ‘change of mind’ were: *(a)* the group’s (and hers) gradually focused discourse on the context of the problem *(b)* the enrichment of her (and the group’s) means of communication by incorporating the ‘pictures-sheet’ in her discourse (e.g. Figure 3) and by drawing on it and *(c)* the researcher’s introduction of an ‘extreme case’ to the task.

Combination of all the group discussions

The ‘Paint’ item was discussed by eight groups, each consisting of three pupils (aged 11 or 12). Eight of these pupils (one in each group) made a ‘constant sum’ error when tested (prior to the discussions) in the ‘Paint’ item but all of them ‘changed their mind’ during the discussions as indicated by their discourse. By combining the basic stages from the discursive paths of each of these eight pupils, a general pattern emerged presented in Table 1.

The general pattern in Table 1 is consistent with Jane’s ‘three-stage’ discursive path and highlights the two basic factors that provide a ‘remedy’ for the ‘constant sum’ method: *(a)* the exploration of the context and *(b)* the various uses of the ‘pictures-sheet’. Furthermore, a third factor that conflicts with ‘additivity’ and influences the pupils towards a multiplicative approach is: *(c)* the introduction of an ‘extreme case’ by the researcher.

Stage	Decision	Explanations	Change in discourse followed:	**Task: PAINT**
1	‘2’ (8 cases)	Adequate, non-multiplicative (7 cases) Numerical, non-multiplicative (1 case)		**Changing from the Constant Sum Error to an ‘adequate strategy’**
2	Reject ‘2’ (8 cases)	Adequate, pre-multiplicative (8 cases)	1. (8 cases) The introduction of the ‘pictures-sheet’ by the researcher and its use, in various ways by the whole group, and then 2. (6 cases) The ‘exploration of the context’ part of the discussion, and then 3. (3 cases) Argumentation from a peer about rejecting the answer ‘2’	
3	Reject‘10’ and adopt ‘14’ (5 cases)	Adequate - multiplicative (5 cases)	1. (5 cases) The introduction by the researcher with the help of the ‘pictures-sheet’ of an extreme case to the task, and then 2. (3 cases) Argumentation from a peer	**Conversion towards a multiplicative method**

Table 1: Changing from the Constant Sum Error to an ‘adequate strategy’.

CONCLUSION

We conclude that pupils who use the ‘constant sum’ method for solving certain ratio tasks could learn to reason more adequately through discussion in small ‘conflict’ groups of children: i.e. groups with conflicting initial responses on the given tasks. Learning, (conceived here as a short term, and perhaps temporary, change of argument or talk in discussion) however, can occur under conditions:

1. The context of the task must be challenging to provoke pupils to negotiate the task contextually first and then seek a solution. It is hypothesized that with this kind of ‘contextual discussion’ pupils can learn to conceptualise a task multiplicatively or, in other words, to ‘model proportionally’. The context of our ‘Paint’ problem (that provokes various interpretations of the ‘way paint is used’) has proven to be exceptionally challenging and effective.

2. A ‘tool’ should be provided that *(a)* makes the context of the task more prominent *(b)* provides a shared means of communication for pupils and *(c)* facilitates their expressions and arguments. This last element is particularly important for young children who do not yet posses adequate mathematical terminology to communicate their thoughts and strategies. The pictorial representation shown in Figure 1 meets these requirements. (Additionally, it affords the crucial argument about (0, 3) that gives an unacceptable red colour instead of orange and brings an apparently successful ‘change of mind’ from additive to multiplicative arguments.)

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