TOWARDS HIGH QUALITY GEOMETRICAL TASKS: REFORMULATION OF A PROOF PROBLEM

Roza Leikin

University of Haifa, Israel

This paper analyses changes in the quality of a mathematical task as a result of its reformulation from a proof mode into an inquiry-based mode. The task is borrowed from the reformulation assignment given to the in-service teachers. The analysis of the task is based on implementation of the task with pre-service teachers. Through the lens of the task implementation I analyze task features that denote its qualities.

BACKGROUND

The quality of mathematical tasks

In their comprehensive analysis of mathematics lessons in US, Germany and Japan, Stigler & Hiebert (1999) pointed out the importance of kind of mathematics that is taught. 'If the content is rich and challenging, it is more likely that the students have the opportunity to learn more mathematics and to learn it more deeply' (p. 57). The researchers consider quality of school mathematics as function of *content elaboration*, *content coherence*, and *making connections* and state that the quality of mathematics at each lesson contributes to development of students' mathematical understanding. Mathematical tasks that the teachers select as well as the settings in which the students are presented with the tasks determine quality of mathematical instruction.

In this paper task quality is defined by the set of four conditions (combined from Polya, 1981, Schoenfeld, 1985; Charles & Lester, 1982). First, the person who performs the task has to be *motivated* to find a solution; Second, the person has to have *no readily available procedures* for finding a solution; Third, the person has to makes an *attempt* and persists to reach a solution; Fourth, the task or a situation have several solving approaches. Obviously, these criteria are relative and subjective with respect to person's problem-solving expertise in a particular field, i.e., a task, which is cognitively demanding for one person may be trivial (or, vice versa, unrewarding) for another.

Teachers' role in teaching high quality mathematics

Inquiry dialog (Wells, 1999) may be seen as a way for increasing the quality of school mathematics. Inquiry tasks usually are challenging, cognitively demanding and allow highly motivated students' work. The students in such an environment conjecture, debate the conjectures, search for explanations and proofs and discuss their preferences regarding different ways of solution. Teachers' awareness (Mason, 1998) of different solving approaches to a problem helps teachers act according to students ideas and be flexible in lesson orchestration (Leikin & Dinur, 2003). Although the importance of a dialogic learning environment is declared, and the teacher's flexibility in the classroom is widely discussed (Brousseau, 1997; Simon, 1997), such a classroom environment remains challenging and vague for teachers. No clear guidelines can be provided for each particular lesson, which is based on students' ideas and conjectures.

The quality of mathematics in any particular classroom depends on teachers' knowledge and beliefs. Teachers' previous experiences often are reflected in their skepticism with regard to changes in the nature and quality of mathematical tasks. Therefore pre-service teacher education is primarily intended to prepare prospective teachers to teach in ways different from those in which they learned as pupils, while in-service education is aimed at developing teachers' proficiency to teach in ways different from those in which they both learned and teach. In this paper I exemplify an implementation of a high quality task in teacher professional development. The analysis is focused on the quality of the task as reflected in problem posing and problem solving procedures in which the teachers were involved.

Supporting quality of mathematical tasks by inquiry in DGE

Nowadays it is rather natural to connect inquiry with computer-based learning environment in general and with dynamic geometry environment (DGE) in particular. This research developed through practice of in-service mathematics teacher education, which was based on the theoretical ideas about implementation of Dynamic Geometry in school mathematics (e.g., Chazan & Yerushalmy, 1998). Teachers' expertise is a crucial issue for an effective implementation of DGE, while one of the characteristics of such an expertise is teachers' ability to formulate powerful mathematics tasks for inquiry in DGE. Many studies explored the role of DGE in teaching-learning processes, namely in concept acquisition, geometrical constructions, proofs, and measurements (e.g., Mariotti, 2002; Jones, 2000). This paper is aimed to analyze changes in the quality of a mathematical task as a result of the adaptation of a proof task for the inquiry in DGE.

THE OBJECTIVES

The main objective of this paper is analysis of the changes in the quality of geometrical that emerge as the result of task adaptation from standard textbook proof task to an inquiry-based problem for work in DGE.

One of mathematics teachers' responses served a starting point for the analysis of the possible outcomes of task re-formulation, which is performed through the lens of problem-solving procedure exposed by 36 pre-service mathematics teachers (PMT) The analysis is based on my observations, written field notes, and videotaped mathematical performances of three groups of PMT.

THE TASK QUALITIES

Meeting the tasks

The task, which is in the focus of this paper, was raised in the in-service course “teaching an inquiry-based mathematics”. In one of the assignments the teachers were asked "to choose a problem from a standard textbook and re-formulate it for an inquiry in DGE". The teachers performed the assignment individually and presented it to other teachers who participated in the course. They explained the re-formulation performed and described the classroom setting in which the re-formulated problem was implemented. One of the teachers (Anat) reformulated a task borrowed from a standard text-book (Goren, 1996):

The original task: *In the isosceles trapezoid ABCD the diagonals are perpendicular (AC⊥BD). Prove, that the altitude of the trapezoid equals to the mid-line joining the mid-points of the two sides of the trapezoid. Prompt: Built the altitude through O [the point of intersection of the diagonals]*

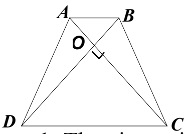
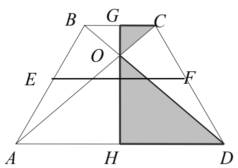


Figure 1: The picture in the textbook

The original task is a prove task, which, according to it placement in the textbook, clearly requires from the students application of the mid-line-of-a-trapezoid theorem. The drawing is presented and the prompt, which is given in the text of the task, simplify the solution and direct it towards one particular solving approach. An intended solution of the original tasks is depicted in Figure 2.



GH-height through *O*, then *OG* and *OH* are the medians, the altitudes and the bisectors in the triangles *BOC* and *AOD*.

Hence, triangles *AHD* and *AGH* are right isosceles triangles: *HD=HO, GC=GO*

Thus, *GH=GO+OH=GC+HD=½BC+½AD=EF*

Figure 2: **A solution of the original task**

Anat reformulated this task as follows:

Anat's problem: *Given an isosceles trapezoid with perpendicular diagonals. Compare the length of the altitude of the trapezoid and the length of the mid-line joining the mid-points of the two sides of the trapezoid?*

Anat's problem included requirement of comparing the two segments instead of proving their equality. The reformulation was based on the opening the task by "hiding" one of the properties of the given geometric figure. So, instead of proving the property, students had to find it, formulate it and prove it. However, this opening was very narrow and purpose directed. Students should, almost immediately, realize that the segments are equal. Additionally, Anat decided to provide her students with detailed guidelines for the construction of the figure in DGE since “usually students spend a lot of time on constructions”. Her guidance included construction of a square and constructing the upper base of the trapezoid by joining the square diagonals inside the square. At this stage of learning the students in her class experienced in using dragging to perform geometrical explorations and the transformed tasks required this type of activity. Anat reported that students conjectured quickly that the segments are equal and proved their conjecture under her guidance.

Raising the task quality through teachers' discussion

Teachers’ discussion on the task presented by Anat focused on the two main issues. First, teachers’ differed in their opinions about the necessity of the *detailed guidelines for the construction*. Some of the teachers, as Anat, preferred providing students with the guidelines since “geometrical constrictions are not part of the curriculum”, and “there is not enough time for this type of activity”. The other group of teachers considered constructing a figure as an integral part of the inquiry tasks in DGE. They told they were happy with any opportunity to "teach geometrical constructions, as they develop students understanding of geometry and students' logical reasoning" Second, the teachers disagreed about the *level of openness of the question*. Some teachers found Anat's problem open enough and confirmed that they would “do the same”. Other teachers tended "to open the task more" both by describing the figure as “an equilateral trapezoid with perpendicular diagonals” and by asking students to explore the figure properties.

One of the teachers, Maya, who was one of the group leaders, suggested a compromise:

Maya's problem: *In the isosceles trapezium ABCD the diagonals are perpendicular (AC⊥BD). Find possible relationships between two midlines of the trapezoid, which join mid-points of the opposite sides of the trapezoid.*

She stated that asking students to find all the properties of the given trapezoid is too vague. She suggested that "replacing the altitude in the original problem by the "second mead-line" makes the problem more elegant" and that changing the word "compare" by "find relationships" opens the question more since "it is clear that the altitude is perpendicular to the midline, however it is not obvious for the two mid-lines".

No consensus was obtained on each of these issues. The teachers liked Maya's problem "for themselves" however were uncertain regarding the task implementation with their students. Teachers' “craft knowledge” (in terms of Kennedy, 2002) embodied in their personal experiences with particular students’ populations, and in their own (mostly limited) experiences with DGE, reflected in their intuitions about what is better for their students and how the tasks re-formulation may be better performed.

Challenging pre-service teachers with Maya's task: The setting

I explored the quality of *Maya's task* with pre-service mathematics teachers [PMT]. The task was probed with three groups of PMT of 12, 10, 12 teachers in each group. All the PMTs had BA in mathematics and were learning first year for the teaching certificate. At each session, that took place at the end of the year, the PMTs worked in DGE in pairs or in small groups of three. Overall 16 small groups (or pairs) were observed and interviewed collectively. I made field notes and reflective notes at each workshop. These notes were discussed with the PMTs during the consecutive meetings in order to confirm research suggestions regarding problem-solving procedures the PMTs encountered.

All the PMTs in three workshops were allowed using Geometrical Supposer (Schwartz, Yerushalmy & Shternberg, 2000) while solving the tasks. Interestingly, all of the PMTs made progress in similar sequences. All the small groups (pairs) of PMTs started with a "freehand drawing" (Chazan & Yerushalmy, 1998). Rather quickly (within 5-7 minutes) most of the participants started construction of the figure so that dragging would preserve the given properties of the trapezoid. Only 1 of 16 small groups of insisted on continuing "freehand drawing" with subsequent correction of the drawing.

When the construction was completed the students carried out measurement of different types and their conjectures were mainly based on the invariants observed while dragging. At the next stage the PMTs proved their conjectures. The next section presents the power of the mathematical tasks by addressing each one of these stages.

No ready-to use procedure: Freehand Drawing

All the PMTs started with drawing a trapezoid and then added diagonals to it. The ways in which the participants created an isosceles trapezoid were similar to those described by Chazan and Yerushalmy (1998). Isosceles trapezoid, which should be constructed in our investigation, had an additional property, i.e., perpendicular diagonals. Thus PMTs constructed the diagonals and “fixed” the angle between them by dragging the trapezoid vertexes. They found themselves “spoiling” the figure by further dragging. They tried to “fix” the figure again and again and sometimes were not able to obtain “exact properties of the figure”.

Obviously this inclination for freehand drawing was borrowed by PMTs from their experience in paper and pencil drawing. To verify this observation the PMTs in two (of three) groups were asked to draw an equilateral trapezoid with perpendicular diagonals on the paper a week after the activity took place. Of 22 participants only 3 started the drawing with diagonals even a week beforehand they discussed in details how to construct the figure. Three teachers started drawing with perpendicular diagonals since the "the drawing will be more precise this way".

Constructing the isosceles trapezoid with perpendicular diagonals in different ways

After realizing that freehand drawing does not allow exploring by dragging PMTs tried to perform "exact construction". This led them to the precise analysis of the properties of an isosceles trapezoid which diagonals are perpendicular. This analysis included thinking about necessary and sufficient conditions of the geometric figure.

At this stage of the work PMTs analyzed "which construction will allow dragging that preserves (a) the quadrilateral as a trapezoid and (b) perpendicularity of the diagonals.

Interestingly in each of the three groups of PMTs at least three different strategies for construction of the trapezoid with perpendicular diagonals were suggested. Figure 2 depicts two of these constructions.

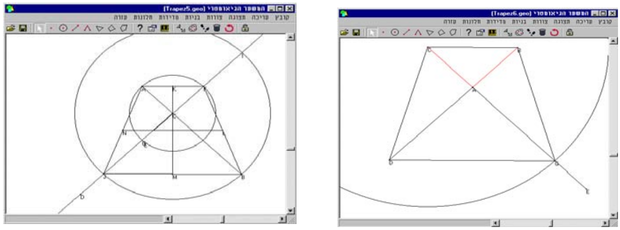


Figure 2: **Two different constructions**

Perpendicular diagonals that are congruent and are divided into two pairs of congruent segments by the intersection point served a sufficient condition for the construction of the given figure. The different constructions included construction of two pairs of congruent segments on the two perpendicular straight lines or completing a right isosceles triangle (the length constrains of the paper do not allow more detailed analysis of the constructions performed). Within each big strategy there were many variations and the teachers were always surprised by the amount of different ways in which different pairs constructed the trapezoid. Note here, that the construction that Anat suggested to her students was not produced by any of the experimental groups.

As mentioned above, only one (of 16) small groups of PMTs argued for sufficiency of freehand drawing. The students in this group were reluctant towards use of DG in teaching school geometry. At the construction stage they "braked and fixed" again and again their drawing and stated that "they may see the regularity". For them the midlines were always "almost perpendicular" and "almost equal" so they come to the conjecture as all other small groups that *in an isosceles trapezoid with perpendicular diagonals the midlines which join mid-points of the opposite sides of the trapezoid are equal each other and perpendicular to each other.*

Proving the conjecture in different ways

Interestingly in each of the 3 groups two different proofs for the conjecture were presented. One of the proofs was similar to one presented in Figure 2. The other proof was based on the construction that the PMTs performed.

Figure 3 depict computer screen in which the internal quadrilateral with vertexes in the midpoints of the given trapezoid is a square since the diagonals in the trapezoid are equal and perpendicular. The midlines that join mid-points of the opposite sides of the trapezoid are equal to each other and perpendicular to each other as the diagonals of the square. This construction-based proof (see Figure 3) usually was found as easier one, more elegant and convincing.

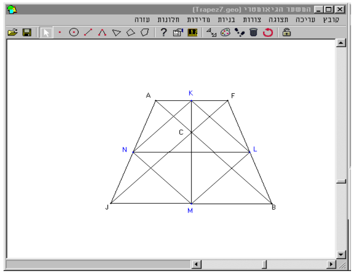


Figure 3: **The construction-based proof**

DISCUSSION AND CONCLUDING REMARKS

In this paper I tried to argue for the raising quality of mathematical tasks when adapting them to inquiry-based learning environment. The quality of the task was defined as depending on the four conditions. The first condition considers motivation for performing the task. As was shown in the paper inquiry problems that fit learners' level stimulate their motivation, like in the case of PMTs shifting from freehand drawing toward a systematic construction. It must be noted that construction procedure which is a part of the inquiry tasks on the one hand deepens analysis of the necessary and sufficient conditions of the given figure and on the other hand complicates the task performance. Based on this feeling of the complexity of the construction tasks many in-service teachers are inclined to provide their students with detailed guidance for the constriction. As it was shown, when performing the inquiry tasks, usually teachers had no readily available procedures for finding a solution. They had to make a certain attempt and persist to reach a solution. I tried to demonstrate that the inquiry tasks contrary to the original proof task had several solving approaches both at the stage of exploring the situation and at the stage of proving the conjectures. The Inquiry procedure seemed to be more connected and elaborated in all the three groups of PMTs. It should be noted that contrary to the Maya's task that is analyzed in the paper, Anat's task is not so distant from the textbook task and does not encompass the same qualities.

Zaslavsky, Chapman and Leikin (2003) suggested that a mathematical task is powerful if it involves dealing with uncertainty and doubt, engaging in multiple approaches to problem solving, identifying mathematical similarities and differences, developing a critical view of the use of educational technology, rethinking mathematics, and learning from students' thinking. The analysis performed in this paper explicitly addresses four of these characteristics. Additionally, one may see rethinking mathematics in teachers' reasoning about geometrical constructions as well as in their exploration of the midlines in quadrilaterals. As the paper presents my own learning from the teachers' thinking I assume that by using this task teachers may learn from students thinking.

Finally I would like to suggest one more condition that may be included into the list of conditions defining mathematical tasks of high quality, namely, the possibility to raise and discuss new mathematical question. I usually continue the mathematical discussion on Maya's task with a question: *Is it possible to inscribe a circle into an isosceles trapezoid with perpendicular diagonals*?

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