A FRAMEWORK FOR DESCRIBING THE PROCESSES THAT UNDERGRADUATES USE TO CONSTRUCT PROOFS

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The purpose of this paper is to offer a framework for categorizing and describing the different types of processes that undergraduates use to construct proofs. Based on 176 observations of undergraduates constructing proofs collected over several studies, I describe three qualitatively different ways that undergraduates use to construct proofs. In the concluding section, I describe the learning that is likely to occur by writing proofs in each of these three ways.

INTRODUCTION

Mathematicians and mathematics educators both agree on the importance of proof and on the necessity for students to develop the skills needed to construct proofs (Blanton, Stylianou, and David, 2003). However, there is also widespread agreement that students have serious difficulties with constructing proofs. Consequently, there has been a great deal of educational research investigating students' proving abilities. Much of the research on proof has examined both the valid and invalid proofs that students produce. The ratio of valid to invalid proofs has been used to provide a measure of students' proof-writing ability (e.g., Senk, 1985) and the invalid student proofs have been used both to classify common student errors and to glean insight into students' conceptions of proof (e.g., Selden and Selden, 1987; Gholamazad, Liljedahl, and Zazkis, 2003). More recently, some researchers have paid less attention to the proofs that students produce and have focused instead on the processes that students use to create those proofs. For instance, Hart (1994) describes the processes that undergraduates use when they are constructing elementary proofs in abstract algebra and illustrates how these processes are influenced by their conceptual understanding. Weber (2001) delineates the processes that undergraduates and mathematicians use to construct proofs about group homomorphisms and demonstrates that undergraduates' proof strategies are often inadequate. Raman (2003) illustrates several approaches that one can take to prove a theorem from calculus and argues that one should base the proof that they are writing on key ideas that they find to be convincing and intuitively meaningful. The purpose of this paper is to further this work by offering a framework that one can use to categorize and describe the processes that undergraduates can use to successfully construct proofs.

RESEARCH CONTEXTS

The framework described in this paper was developed using data from several empirical studies in which I observed eight undergraduates in abstract algebra and six undergraduates in real analysis constructing proofs in their respective domains. (See

Weber (2001, 2002, 2003, 2004) for reports on these studies). There were two abstract algebra studies that were conducted to investigate possible deficiencies in undergraduates' proving processes. The real analysis study was a longitudinal one designed to follow the development of undergraduates' concept understanding and proving abilities. In all three studies, undergraduates were asked to "think aloud" as they proved a collection of statements. After their proof attempts, they were asked to describe why they tried to prove the statement in the way that they did. In total, 14 undergraduates were observed constructing a total of 176 proofs (56 from abstract algebra, 120 from real analysis).

TYPES OF PROOF PRODUCTIONS

Procedural proof productions

In a *procedural proof production*, one attempts to construct a proof by applying a procedure, i.e., a prescribed set of specific steps, that he or she believes will yield a valid proof. It may be the case that the procedure is meaningful to the prover; that is, the prover understands why the successful implementation of the procedure will yield an argument that logically establishes the veracity of the claim to be proven. However, in the studies that I conducted, it was more often the case that undergraduates applied procedures that were not meaningful to them. As a result, they would produce valid proofs, but could not explain what their proofs meant (cf., Weber, 2003). There were many cases where the undergraduates' successful proof attempts consisted simply of mimicking the actions of the teacher or applying a set of steps that they had been told will yield a valid proof.

There were two types of procedures executed by the undergraduates in my studies. The procedure may be an *algorithm*, or a list of steps that were highly mechanical and tied to a specific type of problem (cf., Weber, 2003). An example of an algorithm that can prove identities about summations using induction is given below:

> To prove statements of the form, $\Sigma^{i=n} f(i) = g(n)$, write:
>
> Proof: [Show $f(1) = g(1)$ by direct computation], which establishes the basis case.
>
> Assume $\Sigma^{i=n} f(i) = g(n)$ as your inductive hypothesis.
>
> Then $\Sigma^{i=n+1} f(i) = \Sigma^{i=n} f(i) + f(n+1)$
>
> Which by the inductive hypothesis is equal to $g(n) + f(n+1)$.
>
> [Verify that $g(n) + f(n+1) = g(n+1)$ using algebraic manipulations].
>
> Hence, $\Sigma^{i=n+1} f(i) = g(n+1)$.
>
> Therefore, $\Sigma^{i=n} f(i) = g(n)$ has been proven by mathematical induction.

Many of the students in the real analysis study used an algorithm similar to the one above to prove these types of statements. Note that applying this algorithm requires minimal engagement on the part of the prover; there are few points in the proof where the prover needs to make decisions or reason mathematically. Also note that the only skills and understanding required to write this type of proof are the abilities to evaluate functions for particular variables and to perform algebraic manipulations; an individual without understanding the logic behind inductive proofs or even knowing the meaning of summation might still be able to apply this algorithm.

As a second illustration of a student proving by applying an algorithm, consider Erica's comments as she proved that the sequence {(n-1)/n} converged to 1.

> Erica: I remember doing one like this on our homework. Can I use my notes?
>
> I: Do you think you need to use your notes?
>
> Erica: [laughs] Yeah.
>
> I: OK.
>
> Erica: Yeah, OK I see. You start this proof writing "Let ε greater than zero be given. Let N equal". Now he uses scratchwork over here to find the N. He says, let's see… OK, to show this converges to 1, we… yeah, OK, we start with the absolute value of n minus 1 over n minus 1 and we'll re-write this as (1 – 1/n – 1) which is equal to the absolute value of 1/n… let's see, then he… oh yes, we drop the absolute value sign and say this is 1/n which is less than 1 over big N which is less than ε…

Erica continued her proof by closely relating what she was doing to the proof that the professor had completed in class. She produced what was a fully valid proof. From the proof itself, one could not a lack of understanding on Erica's part. However, subsequent questions by the interviewer revealed that Erica neither understood why her argument was logically valid nor had an accurate understanding of the meaning of the limit of a sequence.

The procedure may also be a *process*, or a shorter list of global qualitative steps that are not highly specified manipulations but rather involved accomplishing a general goal (cf., Weber, 2003). To illustrate a student applying a process, consider the following undergraduate's proof that $n! \geq 2^{n-1}$ for all natural numbers n.

> David: Well the basis case just gives …. 1 is equal to 1. To solve the inductive step, I would have to see how (n+1)! related to n! and how 2^{n+1} related to 2^n. I think that I would approach it in some way of handling the factorial. If I can expand (n+1)! in some way, I can see how it relates to n. If I can see how they are related, I can use my inductive hypothesis.

David went on to construct a valid proof. David's proof by induction involved executing several qualitative steps. For instance, David attempted to write (n+1)! in terms of n! without a clear method specifying how this might be done. He was able to construct a valid proof, even though he had not yet proved statements involving factorials. While David showed considerable skill at writing these types of proofs, a comment made after David constructed the proof revealed that he did not understand why inductive proofs are mathematically valid.

David: And I prove something and I look at it, and I thought, well, you know, it's been proved, but I still don't know that I even agree with it [laughs]. I'm not convinced by my own proof!

Syntactic proof productions

In a *syntactic proof production*, one attempts to write a proof by manipulating correctly stated definitions and other relevant facts in a logically permissible way. In the mathematical community, a syntactic proof production can be colloquially defined as a proof in which all one does is "unpack definitions" and "push symbols". In the mathematics education literature, this type of proof has also been referred to as a purely formal deduction (Vinner, 1991). Two examples, taken from Weber (2001), are given below. In both examples, the undergraduates were asked to prove the following theorem.

Let G and H be groups. G has order pq, where p and q are prime. f is surjective homomorphism from G to H. Prove that G is isomorphic to H or H is abelian.

Jim: Hm… so what do we have here. We have G has order pq… f is a surjective homomorphism from G to H. So… [long pause]… Well, G has order pq so G has an element of order p… by Cauchy's theorem… and likewise G has an element of order q… so since f is a homomorphism, let x be an element of order p, then f(x) would be an element of order p… um, no, an element of order p or order 1…

…

So what did we do on the last problem? We looked at the kernel. So, yeah, these problems tend to build on each other, so what is the kernel going to be here. Um the kernel is going to have size 1, p, or q… oh yeah, or pq. How does that help us? [pause] Um, okay then H is going to have size pq, q, p, or 1. And if H has size p or q, it is cyclic and abelian. And if it has size 1, it is abelian. And if it has size pq? Then it must be isomorphic to G. Why? Um, oh yeah, by f.

Steve: Well, injective… if G and H have the same cardinality, then we are done. Because f is injective. And f is surjective. G is isomorphic to H with the isomorphism being f. OK, so let's suppose their cardinalities are not equal. So we suppose f is not injective. Show H is abelian… OK f is not injective so we can find distinct x and y so that f(x) is equal to f(y). OK, to show abelian, let us choose an h in H. Then f(x)h is equal to f(y)h…

Steve continued to draw logical deductions, such as the fact that $f(x)h = f(x)f(f^{-1}(h)) = f(xf^{-1}(h))$ and also that xy^{-1} would be a member of the kernel of f, but unlike Jim, was unable to construct a proof. Both Jim and Steve's proof attempts appeared to consist entirely of drawing a sequence of logical deductions. Their deductions either involved stating the definition of a mathematical concept or using facts that they knew about the concepts to construct proofs. At no point did either undergraduate consider the semantic meaning of the statements that he was dealing with; they did not, for instance, use visual representations of the groups in question (perhaps because they did not have such representations in their repertoires) and did not see why this statement would be true by considering particular groups of order pq.

Semantic proof production

Mathematical propositions often describe relationships between mathematical objects. In a *semantic proof production*, one first attempts to understand why a statement is true by examining representations (e.g., diagrams) of relevant mathematical objects and then uses this intuitive argument as a basis for constructing a formal proof. In the mathematics education literature, semantic proof productions have also been referred to as proofs following intuitive thought (Vinner, 1991), and are similar to what Raman (2003) calls proofs based on key ideas.

The example below illustrates a semantic proof production in which an undergraduate demonstrates that the sequence {1, 0, 1, 0, 1, 0, …} does not converge.

> Stacey: [After reading the question] Let me first see what this sequence looks like. [Graphs the sequence on a Cartesian plane]. So this doesn't seem to be converging. [Draws a horizontal band between y=0 and y=1]. Yeah, if we make this band thin enough, it's not going to get both of the points… OK, so we'll let epsilon be one-third, which would make the width two-thirds. Then no matter what the limit is, either the 0's will not be in the band or the 1's won't. So no matter what the N is, I can always find an odd or even number bigger than that and the odd sequence term would be 1 and the even one would be 0.

Stacey then wrote a proof using appropriate mathematical notation that reflected what she had just said. In her proof, Stacey did not begin by stating definitions or drawing inferences. Instead, Stacey first tried to understand the claim being made by sketching a graph. In examining this graph, she realized that any epsilon band having a width of less than one would not contain the points. She then wrote (or translated) her intuitive argument into the language of formal mathematics to produce a proof.

DISCUSSION

Learning outcomes of different types of proof productions

Lithner (2003) observes that the way that one solves a problem will affect the nature of what one learns from their problem-solving episode. In this section, I describe how procedural, syntactic, and semantic proof attempts may provide the prover with different levels of conviction and understanding. There are (at least) three important purposes that undergraduates should have when they are constructing proofs in their mathematics courses. Their proof of a statement should convince themselves that the statement is true, promote understanding by explaining why the statement is true, and convince their mathematical community, including their teacher, that the statement is true. In the remainder of the section, I discuss the extent that each type of proof production achieves these three goals.

To most undergraduates, convincing their teacher (and thereby earning satisfactory grades) is typically the most important reason for constructing a proof. Procedural, syntactic, and semantic proof productions can all yield valid proofs; hence all are capable of achieving this goal. Nonetheless, it is worth noting that if undergraduates rely exclusively on procedural or syntactic proof productions, the scope of statements that they can prove may be rather limited (see Weber (2001, 2002) for empirical support of this assertion).

By "convincing oneself that a statement is true", I mean to see why the statement is a logical consequence of previously accepted assertions. Syntactic and semantic proof productions both would convince the prover (in a formal mathematical sense) that the statement is true, but procedural proof productions might not. If the procedure that is being applied is not meaningful to the individual applying it, that individual may produce arguments that he or she does not find convincing. This was illustrated in this paper with David, who could prove a statement by induction, but "still not even be sure that he agrees with it".

Many mathematics educators believe that promoting understanding is the most important reason for introducing proof in the university classroom (e.g., Hanna, 1990; Hersh, 1993). However, both procedural and syntactic proofs may fail to explain to the prover why the statement is true. Many of the undergraduates that I interviewed applied an algorithm similar to that presented early in this paper to verify identities about summations using induction. To apply this algorithm, one does not even need to know the meaning of summation, and certainly does not need to see the identity as establishing an equality between a summation and an equation. Likewise, syntactic proofs may be understood only as symbols obeying logical rules, and not exhibiting relationships between mathematical objects and mathematical structures (cf., Weber, in press). Semantic proofs are based on intuitive representations and will therefore be meaningful to the prover who produces them (Raman, 2003).

This is not to say that undergraduates should not engage in procedural or syntactic proof productions. Having reliable procedures to prove common classes of statements and being able to logically manipulate symbols in a flexible manner are important skills for competent theorem proving, and mathematicians regularly write proofs in this way (Weber, 2001). Further, with reflection, both syntactic and procedural proof productions can serve as the basis for sophisticated learning (c.f., Pinto and Tall, 1999; Weber, 2003). However, there is a danger that if undergraduates *only* write these types of proofs and do not reflect on their proofs or proving processes, then the act of proving may not be effective at promoting understanding.

Types of proof productions by the undergraduates in these studies

Analyzing the proof attempts by the participants in my studies suggests that these undergraduates rarely attempted to construct semantic proofs. Of the 56 proofs attempted by the eight undergraduates in the abstract algebra studies, 46 attempted syntactic proof productions (24 were successful). The other 10 made no progress on the problems and hence could not be categorized. Of the 120 proofs attempted by the six undergraduates in the real analysis course, 48 attempted to produce procedural proofs, 28 syntactic proofs, and only 17 semantic proofs. For the other 27 statements, the undergraduates either engaged in behavior that could not lead to a valid proof (e.g., checked that a general statement held in several instances and presented this as a proof) or made no progress on the problem. Further, in the longitudinal study in real analysis, I also investigated the participants' learning strategies and found that it was relatively rare for these students to reflect on their mathematical work.

Coupled with the preceding analysis, these results suggest that the act of proving may not have been an effective means for these undergraduates to gain understanding. Of course, one cannot determine whether these results are generalizable. That is, one cannot yet claim that most undergraduates rarely produce semantic proofs. It may have been the case that the undergraduates' behavior was due to the idiosyncrasies of their instructor or, perhaps, their behavior was influenced by the nature of the proofs that they were asked to construct. However, if other undergraduates behave in the same way as the undergraduates in this study, then this is a pedagogical problem that should be addressed. Investigations on whether this would be the case would be useful activities for future research.

References:

Blanton, M., Stylianou, D., and David, M. (2003). The nature of scaffolding in undergraduates transition to formal proof. In N. Pateman, B. Daugherty, and J. Zilliox (eds) *Proceedings of the 27th Conference of the International Group for the Psychology of Mathematics Education*, 2, 113-120. Honolulu, HI.

deVilliers, M. D. (1990). The role and function of proof in mathematics. *Pythagoras*, 24, 17-24.

Gholamazad, S., Liljedahl, P., and Zazkis, R. (2003). One line proof: What can go wrong? In N. Pateman, B. Daugherty, and J. Zilliox (eds) *Proceedings of the 27th Conference of the International Group for the Psychology of Mathematics Education*, 2, 437-444. Honolulu, HI.

Hanna, G. (1990). Some pedagogical aspects of proof. *Interchange*, 21, 6-13.

Hart, E.W. (1994). A conceptual analysis of proof-writing performance of expert and novice students in elementary group theory. In J.J. Kaput and E. Dubinsky (eds.), *Research Issues in Undergraduate Mathematics Learning: Preliminary Analysis and Results,* Washington, DC: MAA.

Hersh, R. (1993). Proving is convincing and explaining. *Educational Studies in Mathematics*, 24, 389-399.

Lithner, J. (2003). Students' mathematical reasoning in university textbook exercises. *Educational Studies in Mathematics,* 52, 29-55.

Pinto, M. and Tall, D. (1999). Student constructions of formal theory: giving and extracting meaning. In O. Zaslavsky (Ed.), *Proceedings of the 23rd Conference of PME, Haifa, Israel,* 4, 65– 73.

Raman, M. (2003). Key ideas: What are they and how can they help us understand how people view proof? *Educational Studies in Mathematics*, 52, 319-325.

Selden, A. and Selden, J. (1987). Errors and misconceptions in college level theorem proving. In *Proceedings of the 2nd International Seminar on Misconceptions and Educational Strategies in Science and Mathematics*. Cornell University. Cornell, NY.

Senk, S. How well do students' write geometry proofs? *Mathematics Teacher*, 78, 448-456.

Vinner, S. (1991). The role of definitions in the teaching and learning of mathematics. In D.O. Tall (ed.) *Advanced Mathematical Thinking*, Dordrecht: Kluwer.

Weber, K. (2001). Student difficulty in constructing proofs: The need for strategic knowledge. *Educational Studies in Mathematics*, 48, 101-119.

Weber, K. (2002) Instrumental and relational understanding of advanced mathematical concepts and their role in constructing proofs about group isomorphisms. In *Proceedings of the 2nd International Conference on the Teaching of Mathematics (at the undergraduate level)*. Hersonisoss, Greece.

Weber, K. (2003). A procedural route toward understanding the concept of proof. In N. Pateman, B. Daugherty, and J. Zilliox (eds) *Proceedings of the 27th Conference of the International Group for the Psychology of Mathematics Education*, 4, 395-401. Honolulu, HI.

Weber, K. (2004). Traditional teaching in advanced mathematics courses: A case study of one professor's lectures and proofs in an introductory real analysis course. To appear in *Journal of Mathematical Behavior,* 23(2).

Weber, K. (in press). Investigating and teaching the thought processes used to construct proofs. To appear in *Research in Collegiate Mathematics Education.*