TOWARDS THE EMERGENCE OF CONSTRUCTING MATHEMATICAL MEANINGS

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This paper presents the design and some results of a series of teaching experiments. The design was created to develop a model for everyday maths lessons, that describes the conditions which foster or hinder the construction of new mathematical meanings. The development process includes the students' epistemic processes, their social interactions, the mathematical domain and supporting functions of the teacher.

INTRODUCTION

The theory of interest-dense situations describes how mathematical meanings are constructed in a special kind of situations in everyday maths lessons, so called interest-dense situations. (Bikner-Ahsbahs 2002, 2003a, 2003b) At the same time, these situations characterize favourable conditions which support interest in mathematics. Thus, fostering interest in mathematics and constructing new mathematical meanings mesh together in interest-dense situations. However, new mathematical meanings are not only constructed in interest-dense situations but also during situations which cannot be called interest supporting at all. How can the condition network out of which constructions of new mathematical meanings emerge theoretically be grasped independently whether interest is supported or not?

Based on empirical data the three collective epistemic actions gathering and connecting meanings and structure-seeing were reconstructed during the development of the theory of interest-dense situations. These actions accompanied all processes of constructing new meanings during interest-dense situations and were the basis for building ideal types of epistemic processes. (Bikner-Ahsbahs 2003a, 2003c) Gathering and connecting meanings are activities which provide the tools for students to see mathematical structures whose validity they prove afterwards. The interplay of gathering and connecting meanings on the one hand, and situation directed student and teacher behaviour (see also Williams 2002, 2003), on the other hand, could be regarded as a basic condition for the emergence of new mathematical meanings within interest-dense situations in the sample class (Bikner-Ahsbahs 2003a, 2003b). The question now is whether this condition network can be transferred to other classes, students of another age and other mathematical domains.

Following this question I designed a series of teaching experiments in order to investigate the conditions which support or hinder constructing new mathematical meanings about infinite sets. In the long run my goal is to develop a theory modelling the emergence of mathematical meanings in everyday maths classes.

THEORETICAL BACKGROUND

During the last few years Dreyfus, Hershkowitz and Schwarz have developed a theory to analyse processes of abstractions. (Dreyfus/Hershkowitz/Schwarz 2001; Hershkowitz/Schwarz/Dreyfus 2001) Meanwhile this theory has been adopted in different empirical studies. (Stehlikova 2003; Tsamir/Dreyfus 2002; Tabach/Hershkowitz 2002; Williams 2002, 2003) The authors regard abstraction as a cultural activity leading to the construction of new meanings while reoganizing and restructuring familiar mathematical knowledge into a new structure. Processes of abstraction are driven by needs or motives. (Hershkowitz/Schwarz/Dreyfus 2001). The core concept of this theory is an epistemic action model connecting three epistemic actions: *recognizing*, *building-with* and *constructing*. *Recognizing* refers to recognizing a familiar structure. *Building-with* is seen in the process of combining familiar pieces of knowledge into a new context. It includes recognizing. Processes of restructuring and reorganizing what is recognized, and known to construct new meanings are labelled *constructing*. The authors call this epistemic action model the dynamically nested RBC-model because of the nested and dynamically interwoven characteristic of the epistemic actions.

This model describes an inner-perspective of constructing new meanings dependent on the situational conditions, the biographical background and the interactional possibilities. Whether or not utterances indicate recognizing, building-with or constructing is due to the students' biography and their individual abilities. Since the development of this theory was a process of abstraction itself the same should be true for the RBC-model. The authors claim the RBC-model to be a suitable instrument for analyses of learning processes that fit the definition. Indeed, available data from which the authors extracted their theory only show teaching experiments with didactical designs for the construction of new knowledge that step by step and systematically builds upon previously constructed structures. Thus, the RBC-model is especially useful for analyses of the just described construction processes. To what extent it is suitable for the analyses of more open situations with spontaneous constructions of meanings is an open question at present.

The model of collective epistemic actions, gathering and connecting meanings and structure-seeing has been extracted from data describing interest-dense, hence, more open situations with a wide range of opportunities for the student to spontaneously construct new meanings. This model characterizes an outer-perspective because it is not really necessary to know details about the students' learning biographies in order to decide what kind of action occurs. Weather or not a structure is new, the group usually indicates during the interactions. To what extent both models can be combined into an integrative model that characterizes the emergence and the process of constructing new mathematical meanings is the key question in this project.

Nearly all data that document processes of the emergence of new knowledge show: If teachers participate in these situations they mould them to a certain extent. Therefore an integrative model of epistemic actions should include the supportive function of the teacher. Although teachers are included in investigations with both models concerning social interactions they are not integrated into the models directly. This has yet to be done with an integrative model of epistemic actions.

METHODICAL AND METHODOLOGICAL REFLECTIONS

Comparing infinite sets as an initial activity

Tsamir and Dreyfus have reconstructed the process of constructing meanings of Ben, a talented student of grade ten. During the interview they provoke different representations of countable infinite sets of numbers and evoke a contradiction about their size comparing the natural numbers. This contradiction induces Ben to reflect on the contradiction itself. This way he constructs meanings on different levels: on the mathematical content level and on a more reflective level of building mathematical theories (Tsamir/Dreyfus 2002). The central activity in these teaching experiments is comparing sizes of infinite sets. Results from this case study may be integrated into the development process of the integrative model if the design of gathering data is created in a similar way to that of the two authors.

I have taken up the basic idea of comparing infinite sets. This idea is included in the creation of a series of experiments. The probands are students of grade nine. The task in the teaching experiment consists of a preparation and core task.

Preparation task: One card after another is uncovered from a pile of cards. The cards show the natural numbers as a sequence to 7. Then cards from another pile are uncovered. This time the cards show the squares of the natural numbers to 49 (figure 1). The rest of the two piles are put beside the two sequences.

1	2	3	4	5	6	7	pile of cards
1	4	9	16	25	36	49	pile of cards

Figure 1: Comparing infinite sets of numbers using stacks of cards

Imagine you could write all natural numbers and all square numbers on cards. Which pile of cards would need more cards? Why? For assistance further piles are prepared. Every pile shows different numbers for each pair of cards taken. Each pile is presented one by one but only if it seems necessary.

Core task: The next representation schema made up of fractions is presented and explained (figure 2). The first row consists of all fractions with the denominator 1, the second one with denominator 2, and so on. Are there as many natural numbers as there are fractions? Why?

$$
\begin{array}{ccccc}
\cdots & \cdots & \cdots & \cdots & \\
\frac{1}{4} & \frac{2}{4} & \frac{3}{4} & \frac{4}{4} & \cdots \\
\frac{1}{3} & \frac{2}{3} & \frac{3}{3} & \frac{4}{2} & \cdots \\
\frac{1}{2} & \frac{2}{2} & \frac{3}{2} & \frac{4}{2} & \cdots \\
\frac{1}{1} & \frac{2}{1} & \frac{3}{1} & \frac{4}{1} & \cdots
\end{array}
$$

Figure 2: Representation schema for all fractions

The series of experiments – an overview

The development of insight during this research process is conceptualized as an iterative process. The series of experiments consists of teaching experiments. It starts with laboratory experiments and it finishes with field experiments. Each step consists of the conduction of the experiment, gathering and analysing all the data of the step at the current experimental stage. After extracting hypotheses from the analyses the next experiment is prepared and carried out. Using the concept of theoretical sampling from the grounded theory (Strauss 1994) sampling is theoretically based on previous insights in order to prove hypotheses and to continue the theoretical trace carried out before. The design is carefully adapted according to the choice of probands and a theory based variation of the didactical and interactional conditions.

To get a deeper insight into different theoretical perspectives analyses are adapted to the principle of triangulation. Data from each experiment are analysed from three different theoretical perspectives:

- From an epistemological perspective
- From a situational perspective
- From an individual perspective

In the first perspective the two epistemic action models are used, in the second and the third perspective the situational conditions are analysed using background theories of social interactions with a focus on the social environment, on the one hand, and the students' and the teacher's behaviour on the other.

The first experiment as an endurance test

The series of experiments start with so called endurance experiments. This means:

- The design will prove the capacity of the two epistemic models to some extent.
- The tasks are not separated into systematically assembled steps but are arranged in an open way to give the students as much leeway as possible for spontaneous and unexpected constructions of meanings.
- The interviewer acts in two roles, the role of a teacher and the role of a test administrator. She restricts herself as much as possible to the role of the researcher.

She presents material and ensures that the process of constructing meanings continues until the problems are solved. To shape a friendly atmosphere she tries to understand the students' utterances from their view. At the same time she avoids as far as is possible giving assistance concerning the content, emotional support or acknowledgement.

The teaching experiments begin with students who are interested in mathematics and voluntarily solve mathematical problems, for the probands should be able to cope with endurance situations and still reach the level of constructing new mathematical meanings. In the course of the series of experiments more and more, not necessarily interested students will be included in the sample.

Three kinds of data: informing – acting – commenting

The three-step-design of each experiment consists of three phases assembling different modes of data, which are included in the analysis of each iteration step (Busse/Borromeo Ferri 2003). The phases are video recorded. The first phase lasts about five or ten minutes, the second about an hour and the third has no time limit:

- Phase 1: The probands are asked what infinte means to them (informing).
- Phase 2: The probands work on the tasks (*acting*).
- Phase 3: The probands and the researcher watch the video records of the second phase. The students stop the film and comment on the situations at important points. They are asked to say how they experienced the process of working on the problems (c*ommenting*).

SOME DATA, RESULTS, AND CONCLUSIONS

The transcipt of the first experiment is worked out but the analyses are not yet finished. Nevertheless, they already show interesting tendencies. Both epistemic action models seem to complement each other. The model of collective epistemic actions describes collective gathering and connecting activities where familiar structures are recognized and which are the basis for building-with processes. If, like in our case, no mathematical knowledge about infinite sets is available the attempt to link the new context and previously constructed structure seems to be obvious. The students' metaphors link known mathematical structures as source systems with the new context to enlarge the range of possible actions with new objects.

Example 1: Toni says *infinite* is a number that is bigger than any other number. This idea is elaborated by Toni and his friend Robin in common: "this number exists on a theoretical basis only", "the number infinite (infinity) is a theoretical assumption which never can be proved", "probably we have a theoretical number that probably could exist from the theoretical view but that does not really come out anywhere in practice". Therefore infinite is conceived as a theoretical number but not as a practical one. The metaphor *infinite is a number* enlarges action possibilities and broadens the students' number concepts. Along with this and during the further going second phase infinite is used as a counting number with special rules: $\infty:3 = \infty$, $\infty - 2 = \infty$, $(\infty - 2):2 = \infty$, $(\infty - 2):4 = \infty$, ... More metaphors like *infinite as a process, infinite as one point infinitely far away, infinite as a term* enable the students to use further reaching considerations and new kinds of acting.

Thus, possible considerations of infinite are gathered. They are implicitly shown in the students' actions and in their usage of metaphors. Metaphors link familiar sign systems with the potentially new sign system. They provide a choice for recognizing familiar mathematical objects which are combined and worked out on a trial basis. If Robin and Toni reach contradictions or limits during activities of building-with they restart gathering and connecting. This way they gain new tools and thus, an extended basis for recognizing and building-with.

The following hypothesis is derived from the preparation task and can be already proven in the analyses of the core task and of some data from other investigations:

> *Rich and fruitful problem based gathering and connecting activities are basic activities for the emergence of recognizing and building-with actions.*

Recognizing and building-with activities do not always lead to constructions. I will refer to a scene now where the two boys succeed in constructing new meanings. Through this I will show how interaction processes contribute to the construction of new meanings and will use this insight to deduce an important support function of teacher.

Example 2: During the preparation task Robin and Toni compare the size of the set of natural numbers with the size of the set of natural numbers beginning with 3. In the remark "you may balance it out again and again" Robin uses a the metaphor of *balancing out something*. By this he seems to explain to himself that the sizes of the sets of natural numbers beginning with 1 and beginning with 3 are equal. However, Robin does not work his idea out, not even when the interviewer asks him to do so. Instead, Toni elaborates it:

> Toni: yeah you can if you just take these two number rows (points at 1,2,3,4, ... and ,then at 3,4,5,6, ...) let's assume the infinity goes on and on and it is now at ,ten then this is at ten too (points at 3,4,5,6, ...) but includes two less but then it goes on until twelve. This is balanced OUT again. If that one goes on until twelve (points at 1,2,3,4, …) then that has two less. (points at 3,4,5,…)

Although the boys had never used an inductive proof before, Toni's way of argumentation is based on the idea of it and the preconception of an infinite process. He focuses on the first ten steps of the infinite process of uncovering cards written with natural numbers whose last number is ten. The segment of the comparison set of natural numbers beginning with 3 and finishing with the number 10 has two numbers less. The idea of an infinite process allows the supplement of the next two numbers to 12: "that is balanced OUT again" Toni says. Now he focuses on the segment of natural numbers to 12 which has two numbers more than the comparison segment. Continuing his thought process Toni lengthens the segment of natural numbers beginning with 3 to the number 14. Afterwards this thought process is worked out in more detail.

Robin has used – probably in an unconscious way – a metaphor which he is not able to work out. Obviously Toni recognizes its potential for reasoning and constructs a way of arguing that confirms the validity of the statement that both sets have the same size.

Example 3: A similar phenomenon can be observed during the core task. Different attempts to construct all fractions as a sequence have failed. The central obstacle seems to be the order type of the set of fractions: The set of positive fractions does not have a minimal element and is dense. All of a sudden Toni starts to count beginning with $\frac{1}{1}$ ("one oneth") while moving his fingers from one fraction to another in a diagonal counting pattern (figure 2). Despite the request of the interviewer Toni does not react. He does not seem to be aware of what he is doing. A bit later Robin takes up the way Toni has counted and shows that the set of natural numbers and the set of fractions are equivalent.

The *metaphor of balancing out something* as well as the counting finger motion is a subconsciously used sign. These signs transform the situations into situations with an increased range of action. Obviously more can be expressed, said or shown than consciousness may grasp explicitly. The metaphor of balancing out something and the counting finger motion appear as an *offer of work* for the other student who now has extended possibilities to act. Exactly this idea to transform the situation through *offers of work* which can extend action possibilities underlines the importance of the teacher's support function. Re-analyses of data about interest-dense situations confirm:

> *Teachers are able to transform situations constructively and these transformations of situations support the construction of new mathematical meanings even if students are not especially interested in mathematics.*

This is shown by the use of *metaphors* and *offers of work* like motions, diagrams modelling student behaviour, or patterns of sketch-program actions where the teacher takes up a student's action sketch and transforms it into an action program. But how can support functions be included into an integrated model for the emergence of new mathematical meanings? This is an open question at present that has yet to be investigated.

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