INTEGRATING CAS CALCULATORS INTO MATHEMATICS LEARNING: PARTNERSHIP ISSUES

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Computer algebra system (CAS) calculators are becoming increasingly common in schools and universities. While they offer quite sophisticated mathematical capability to teachers and students, it is not clear at present how they may best be employed. In particular their integration into students' learning and problem-solving remains an issue. In this paper we address this issue through the lens of a study which considered the introduction of the TI-89 CAS calculator to students about to enter university. We describe a number of different aspects of the partnership they formed with the calculator as they began the process of instrumentation of the CAS in their learning.

BACKGROUND

Since Heid's (1988) groundbreaking study, research on the use of computer algebra system (CAS) calculators in the learning of mathematics has tended to concentrate on specific content, such as aspects of algebra or calculus. However, until more recently there has been less emphasis on the processes by which students (and teachers) decide whether to use CAS, and if so, how and when to use it in learning. This is a major area of study since the process of integration of CAS into learning is not a minor consideration but the formation of suitable schemes involves numerous decisions and interactions (Thomas, 2001) with the technology.

A number of studies have described how technological tools may be employed in qualitatively different ways. For example, Doerr and Zangor (2000) have listed property investigation; computational; transformational; data collection and analysis; visualizing; and checking as ways in which technology may be useful. Goos, Galbraith, Renshaw and Geiger (2000) describe a hierarchy of technology interactions, where the student may be subservient to the technology, the technology can be a replacement for pen and paper, can be a partner in explorations, or an extension of self, integrated into mathematical working. A particularly useful approach, based on the ideas of Rabardel (1995), distinguishes between the use of technology as a *tool* and as an *instrument*. Transforming a CAS tool into an instrument involves actions and decisions based on the adapting it to a particular task via a consideration of what it can do and how it might do it. Trouche (2000) and Guin and Trouche (1999) have explained that this instrumentation process and the conceptualisation process are dependent on each other and that for instrumentation to occur classroom activity must be directed at particular conceptions. The study of Drijvers and Herwaarden (2000) concluded that both the technical and conceptual aspects of instrumentation need explicit attention, and that integration of CAS with pen and paper substitution and isolation techniques will lead to improved results.

Theories of instrumentation (e.g., Lagrange, 1999a) stress that each student who uses CAS has to work out its role in their learning. They have to learn to decide what CAS is useful for, and what might be better done by hand, and how to integrate the two (Thomas, Monaghan, & Pierce, in press). When controlling the machine they

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have to be aware of possibilities and constraints, of possible differences between mathematical and CAS functioning, of symbolic notations and internal algorithms. Then there is the issue of monitoring the operation of the CAS (e.g., the syntax and semantics of the input/output, the algebraic expectation, etc), and the difficulties of navigating between screens and between menu operations.

These issues have given rise to some problems with CAS use. For example, Monaghan, Sun and Tall (1994) record how, for some students, CAS can become a mere button pushing process that obscured deeper understanding. In turn, Hunter *et al.* (1993) found that not only did CAS not motivate students, but they became dependent on it, performing worse than a control group on factorising and expanding when not using CAS. A similar outcome is recorded by Hong, Thomas and Kiernan (2000) who showed weaker students becoming reliant on CAS as a problem-solving support, resulting in a negative effect on their learning. On the other hand, Drijvers (2000) encourages taking a positive approach to the obstacles that students may encounter during instrumentation, and how these may be overcome. It seems to us that the key lies in the students' ability to engage in instrumentation, to form a partnership with the CAS whereby they are comfortable with integrating it into their learning, problem-solving and mathematical practice. This paper addresses merely the genesis of this process, looking at the ways in which a small group of students begin choosing to use CAS in their mathematical work.

METHOD

A one week workshop was arranged for students taking a standard first year mathematics course at The University of Auckland which uses the TI–89 calculator. Eight students enrolled for the workshop, 5 females and 3 males, aged 18 to 26, with the exception of two older females, who were 51 and 55. None of the 8 students in the study had ever used a CAS calculator before, but all except one had used a scientific calculator. Each student was given their own TI–89 CAS calculator during the workshop which they kept for the whole week. The workshop covered basic functional aspects of the TI–89 along with use of the CAS calculator's more advanced features when solving problems in calculus and linear algebra. There was also discussion on the learning of core mathematical concepts using the calculators.

The second named researcher taught on the workshop for five two-hour sessions, demonstrating some points using a viewscreen while students followed and copied her working onto their own calculator. Afterwards the students spent the rest of the time working on problems and tackling exercises as a group, while the researcher circulated and assisted with any difficulties. The students were given a pre–test prior to the workshop to ascertain their knowledge of calculus and algebra, and four different post–tests during the workshop, one after each two-hour section based on that day's material. The tests (of 5 to 7 questions) comprised procedural and conceptual questions (see the results section for some of the questions).

One of the aspects of the students' work we were particularly interested in, and which forms the focus of this paper, was both the manner and the timing of the students' CAS calculator use. In order to have some idea of the use they were making of the technology we asked them to mark it by putting a ◆ symbol alongside the point at which they used the TI-89 calculator to help them answer a question, and to give some idea of how or why they used it. The analysis which follows comprises a discussion of these uses.

RESULTS

Since none of the students had used a CAS calculator before, they were all at the genesis of instrumentation of this particular tool, beginning to form the partnership necessary to integrate its use into their mathematics learning and problem solving. What our analysis of the data revealed was a number of qualitatively different categories of CAS use, each of which is considered below.

Direct Use of CAS for Straightforward and Complex Procedures

As Thomas (2001) has described, interactions with CAS representations can be procedural or conceptual. A number of students chose simply to use the CAS to perform direct procedural calculations (i.e. a single command mapping directly to the mathematical operation) instead of doing them by hand. Sometimes they did so when the calculation was relatively straightforward and probably could have been done by hand, and sometimes when it appeared that the calculation was either too long or too complex for them to do it by hand. In Figure 1 we see examples of the former type for a limit question. These two students did not do the limit question in the pre-test and so may not have been able to do these by hand. In Figure 1 student 8 writes the ◆ symbol alongside her solution to show that she had simply entered the limits, while student 2 shows what she did by re-writing the command entered into the calculator (the F3 is the menu selected and 3 the item in the menu).

◆ (i) $\lim_{x\to\frac{3}{2}} \frac{4x^2-9}{2x-3} = 6$

Student 8

◆ (ii) $\lim_{x\to+\infty} \frac{8x-1}{2x+6} = 4$

F3 - 3

Limit (8x-1)/(2x+6), x, ∞)

= 4

Student 2

Figure 1. Direct procedural use of the CAS, replacing by-hand working for accessible calculations.

Figure 2 shows an example of direct procedural use of the CAS when the by-hand procedure may be too complex for the student to carry it out. Here student 8 has used the 'Solve' function of the CAS to solve an equation where the variable *x* is the index. The lack of intermediate steps, the brackets around the 3s, and the use of the ◆ shows the use of a direct CAS command.

◆ (iii) $e^{2x} = 3^{2x-4}$ (give the 'exact' solution).

$x = \frac{2\ln(3)}{\ln(3)-1}$

Figure 2. Direct procedural use of the CAS, replacing by-hand working for complex calculations.

Using CAS to Check Procedural By-Hand Work

It was expected that our students would use CAS to check pen and paper working. Figure 3 shows such a strategy employed in question 2 from the second post–test, "Find the gradient of the tangent to the graph of $y = 3x^3 - 5x^2 + 7x - 9$ at x=1". We see student 3's by-hand working to the left, finding the derivative of y and substituting x = 1, giving 9–10+7, or 6. On the right we see where he has checked with the CAS calculator whether the answer is correct. Here again the process can be carried out directly by employing a single command to differentiate the function with respect to x (using d(3x^3–5x^2+7x–9, x)) as well as calculating the value of the derivative at the point where x=1 (using |x=1).

$9x^2 - 10x + 7$ 6 d(3x^3-5x^2 +7x -9, x)|x=1

$9 - 10 + 7$

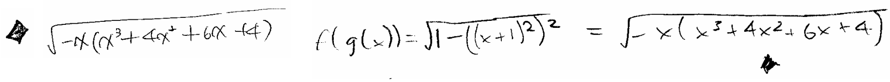
Figure 3. Student 3's direct use of CAS for checking by-hand working.

Direct CAS Use Within a Mathematical Process

Another category of CAS calculator use at first seemed to be another direct use. However, further consideration showed that students were not simply using the CAS to perform the whole calculation, as seen above, but there was a partnership evolving with CAS assigned a defined role within the overall solution process. In some questions students appeared to reach a point where the mental load required to keep the mathematical concepts in mind along with the overall process appeared to be sufficiently large that the they decided to resort to the CAS to handle a procedural aspect, possibly to reduce cognitive load (Sweller, 1994). One example occurred in question 4 of the first post-test:

If $f(x) = \sqrt{1 - x^2}$ and $g(x) = (x+1)^2$, find $f \circ g(x)$.

In order to use CAS for this students first have to undertake some preliminary CAS activity, requiring them to define two functions, f and g, and to understand the need to enter $f(g(x))$ into the calculator for the composite function. Understanding the composite function by-hand working would produce $\sqrt{1-((x+1)^2)^2} = \sqrt{1-(x+1)^4}$, but 4 of the 8 students chose CAS use, possibly either because the procedure was too complex, or they feared errors. They obtained answers like those shown in Figure 4a. The extent of their conceptual understanding of the concept of a composite function remains unclear, and we cannot decide from the answers whether or not they used the CAS to avoid cognitive overload. Other students clearly understood the conceptual part of the composite function and produced $\sqrt{1-((x+1)^2)^2}$ by hand. However when student 6 (see Figure 4b) decided to take this further and simplify it she chose the CAS to do so, presenting her working in a way that we can see this. Again she is using CAS within the mathematical process, and decisions on when and how to do so form a significant part of the instrumentation process.



4a 4b

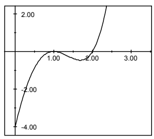
Figure 4. Students 7 and 6's answers to question 4 using CAS commands.

Using CAS to carry out complex procedural calculations can raise difficulties, for example through notation constraints. This is exemplified by the working of student 5 (see Figure 5), who was challenged by the format of the answer provided by the CAS. An observation of the surface structure (Thomas & Hong, 2001) of the function under the square root sign leads him to believe that the function is negative, and hence his comment that there is "No real result". He is not able to rationalise the root and negative signs with the domain of x in order to consider whether the function can be positive for some x values. This demonstrates that the format in which CAS gives answers can lead to problems which challenge understanding.

Figure 5. Student 5 uses the CAS procedurally and meets a challenge with the answer format.

Question 2 of post-test 3, asked for a sketch of an antiderivative function:

The graph of a function is shown in the figure… Make a rough sketch of an antiderivative function F alongside, given that $F(0)=0$.



In response to this Figure 9 shows the working of student 3. He has first moved from the given graphical representation to an algebraic representation, working by hand to get $f(x)=a(x-1)^2(x-2)$ and then using $f(0)=-4$ to find the value of a. At this point in the solution process he resorted to the CAS to integrate and find an antiderivative function. He then moved back to the graph mode, sketching the graph of this function by hand (it appears), rather than using CAS and copying the graph. Again it appears that complex interplay between known algebraic schemes, cognitive load, and ability to perform procedures are driving decisions about CAS use.

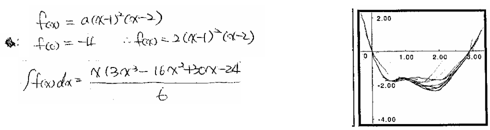


Figure 6. Student 3 uses the CAS procedurally within a mathematical solution.

Using CAS to Investigate Conceptual Ideas

All of the above have involved direct use of CAS, a single procedural command calculating or evaluating some function or expression, which forms the answer. However, in some of the questions there was more interaction with mathematical concepts. For example, question 3 of the first post-test asked:

If $f(x)=\begin{cases} x^2+2 & \text{for} \quad x<1 \\ x & \text{for} \quad x \geq 1 \end{cases}$, then using properties of limits, find out whether or not *f(x)* is continuous at $x=1$.

Rather than simply asking for a result this question was assessing the concept of continuity. Students 2 and 4 (see Figure 7), and others, integrated the CAS into their approach, deciding to use it to find the left and right limits of *f(x)* at *x*=1. While the CAS performs a procedure each time, to embark on this method they needed to know that these limits were relevant and fundamental to the definition of continuity. Student 2 found that CAS would not give the limit at *x*=1 directly (undefined), but then only found the right-hand limit. Student 4 has answered the question completely (CAS use again shown by the ◆), combining conceptual knowledge with CAS procedural results.

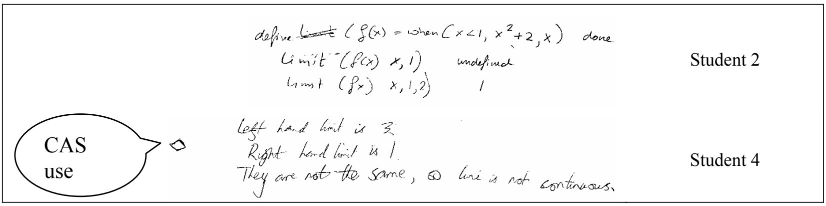


Figure 7. Use of CAS procedures within a test of the definition of continuity.

Another question where integrating the CAS into conceptual thinking seems to have been useful to students was question 6 of post-test 1. It asked:

Let $f(x)=\dfrac{x^2-6x+8}{x-4}$. Sketch the graph of *f(x)*. Can you explain why the graph has this form?

There were several approaches possible here using the CAS. Some students chose to use the CAS immediately to draw the graph of the function. In Figure 8 we see that student 5 indicates that he used the CAS to draw the graph. While the discontinuity at *x*=4 is not shown on the CAS screen, he is able to combine the graph with his understanding of the function to state that "At *x*=4, *y* is undefined."

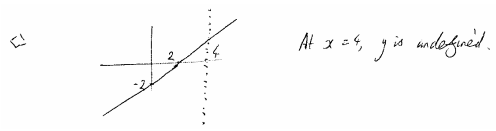
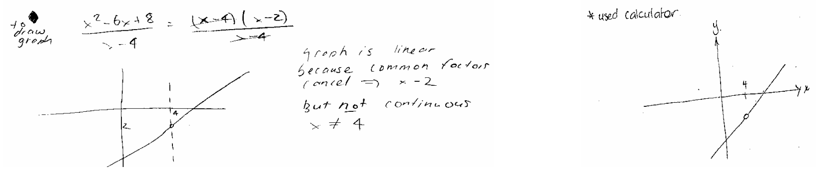


Figure 8. Combining CAS with understanding to answer a question.

In contrast, student 6 (see Figure 9a), appears to have chosen to simplify the rational function by hand and then use the CAS to draw the graph, even though it reduced to a linear function (she shows the by-hand working and explicitly states the use of CAS ‘to draw [the] graph’). Having obtained the graph she then was able to combine her understanding of rational functions to show the ‘missing’ point at x=4 and to say that the function was “<u>not</u> continuous $x \neq 4$.” Unfortunately the line at $x = 4$ is on the wrong side of the x-intercept, giving y as –2 instead of +2. This, along with student 4’s answer in Figure 9b, illustrates that since the CAS graphs do not show the scale values on the axes, one of the necessities of successful integration is care when transferring attention from CAS mode to by-hand working.



9a 9b

Figure 9. Integrating CAS into a conceptual approach to a question.

This difficulty could have been surmounted by recourse to the table mode of the CAS, but it appears that, at this early stage, none of the students had a sufficiently developed instrumentation of the CAS to consider this.

CONCLUSION

In this paper we have considered the instrumentation of the CAS calculator as students begin to use it in solving mathematics problems. The results are consistent with the view that such instrumentation is not a short, easy process, but rather its development takes time. Students have to form the techniques and utilisation schemes (Lagrange, 1999b) required. Our research showed that the students were more likely at first to learn the use of buttons and menus for entering direct single procedures into the CAS, often to check their by-hand working. The process of making decisions about when and how to use the CAS in longer or more difficult mathematical problem solving raises obstacles that come later (Drijvers, 2000). This process may begin with procedural use within a question and then later proceed to cases where the CAS is used to explore conceptual ideas, using several procedures and representations.

The specific categories of CAS use that we have identified are:

- Performing a direct, straightforward procedure,
- Checking of procedural by-hand work,
- Performing a direct complex procedure, for ease of use, or because the procedure is too difficult by hand,
- Performing a procedure within a more complex process, possibly to reduce cognitive load,
- Investigating a conceptual idea.

Of course we have simply made a start in analysing types of usage when CAS is integrated into mathematical work. The final category above is where much of the value of instrumentation lies, and it will no doubt yield a number of subcategories of its own. In future research we intend to provide the students with richer problem solving activities in order to investigate the nature of the thinking elicited, and the decisions which lead to integration of CAS. It is in these kinds of situations that we believe a real partnership with CAS will emerge.

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