MATHEMATICS TEACHERS' KNOWLEDGE BASE: PRELIMINARY RESULTS

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Student learning depends on the teacher's actions, which are, in turn, dependent on the teacher's knowledge base—defined here by three components: knowledge of mathematics content, knowledge of student epistemology, and knowledge of pedagogy. The purpose of this study is to construct models for teachers' knowledge base and for their development in an on-site professional development project.

THEORETICAL FRAMEWORK

Building on Shulman's (1986, 1987) work and consistent with current views (e.g., Cohen & Ball, 1999, 2000), Harel (1993) suggested that three interrelated critical components define teachers' *knowledge base*: (a) knowledge of mathematics content, (b) knowledge of student epistemology, and (c) knowledge of pedagogy:

Knowledge of mathematics content refers to the breadth and, more importantly, the depth of the mathematics knowledge possessed by the teacher, particularly, their *ways of understanding* and *ways of thinking*—terms to be defined in the sequel. The content knowledge is the cornerstone of teaching for it affects both what the teachers teach and how they teach it.

Knowledge of student epistemology refers to teachers' understanding of fundamental psychological principles of learning. This includes knowledge on the construction of new concepts.

Knowledge of pedagogy refers to teachers' understanding of how to teach in accordance with these principles. This includes an understanding of how to assess both students' existing and potential knowledge, how to utilize assessment to pose problems that stimulate students' intellectual curiosity, how to promote desirable ways of understanding and ways of thinking, and how to help students solidify the knowledge they have constructed.

Ways of Understanding and Ways of Thinking

Harel (1998) distinguished between these two categories of knowledge—*ways of understanding* and *ways of thinking*—upon which have been elaborated by Harel and Sowder (in press): Generally speaking, a way of understanding (*WoU*) refers to either a student's (a) meaning/interpretation of a term or sentence, (b) solution to a problem, or (c) justification to validate or refute a proposition. A way of thinking (*WoT*) refers to "what governs one's ways of understanding, and thus expresses reasoning that is not specific to one particular situation but to a multitude of situations." Harel and Sowder (in press) classified WoT into three categories:

problem-solving approaches, proof schemes, and beliefs about mathematics. The three categories are not mutually exclusive.

Problem-solving approaches: Examples of problem-solving approaches include "look for a simpler problem," "examine specific cases," and "draw a diagram." Unfortunately, some teachers, in attempts to improve problem-solving performance with students, advocate problem-solving approaches that can render sense-making in mathematics unnecessary. "Look for a key word in the problem statement" and "look for relevant relationships among quantities based on their units" are examples of such approaches.

Proof Schemes: Proving is defined in Harel and Sowder (1998) as the process employed by a person to remove or create doubts about the truth of an observation. A distinction is made between two processes of proving: *ascertaining* and *persuading*. "Ascertaining is a process an individual employs to remove her or his own doubts about the truth of an observation. Persuading is a process an individual employs to remove others' doubts about the truth of an observation" (Harel & Sowder, 1998, p. 241) Thus, a person's *proof-scheme* consists of what constitutes ascertaining and persuading for that person. Harel and Sowder provided a taxonomy of proof scheme, which was later refined in Harel (in press).

Beliefs: Here beliefs refer to the teacher's views about the nature of mathematics, of knowing mathematics, and of learning mathematics. Examples of beliefs are "mathematics is a web of interrelated concepts and procedures" and "understanding mathematical concepts is more powerful and more generative than remembering mathematical procedures" (Ambrose et. al., 2003, p. 33). These are obviously desirable beliefs. Examples of undesirable beliefs are "formal mathematics has little or nothing to do with real thinking or problem solving" and "only geniuses are capable of discovering or creating mathematics" (Schoenfeld, 1985, p. 43). On the one hand, one's beliefs influence the way one interprets a situation, understands a mathematical statement, and approaches a problem. On the other hand, one's beliefs evolve as one learns and does mathematics.

METHOD

A two-year, on-site professional development research project is underway to study the evolution of teachers' knowledge base. The site is a public middle/high school that offers an intensive college preparatory education for low-income student populations. The school adopts block schedule in which each class meets five times (four 100-minutes and one 75-minute lessons) in a two-week period. Three teachers have participated in this project.

One class of each teacher was observed by us once or twice each week. We then met for 30-45 minutes with the teacher a few days after each observation to discuss the teacher's goals for the lesson and help the teacher reflect on the activities observed during the lesson. So far, we have conducted a total of 14 such observation-conversation pairs per teacher. The teachers understand that the observations are not to evaluate their teaching ability but a source for them and for us to learn about the learning and teaching of mathematics. All three teachers were eager to participate in this project and have enthusiastically shared their ideas with us.

The classroom lessons (except the first one or two lessons) are videotaped and the conversations are audio-taped. During the teacher-researcher conversation, a teacher shares his (or her) objectives and rationales for his teaching actions, his thoughts about students' WoU and WoT, and her or his plans for subsequent lessons. We pose mathematical and/or didactical situations to test our hypotheses about specific aspects of the teachers' knowledge base.

At present, we have analyzed the first three observation-conversation pairs for one Algebra II teacher whom we call Bud (a pseudo-name). The analysis includes dividing each observation/conversation into segments (roughly speaking a segment is a self-contained episode of an observed classroom activity or of a dialogue in a conversation) and analyzing each segment with the previous observation-conversation pairs in mind.

RESULTS AND DISCUSSION: FOCUS ON THE CONCEPT OF FUNCTION

Conversations

The following is an excerpt from a dialogue between the first author (H) and Bud (B). It reveals certain aspects of Bud's way of understanding the concept of function:

> H: Do you think that they [the students] know what function is?
>
> B: Not really. [My] last year class is an indication that they don't really understand what a function is?
>
> H: … What is for you … understanding function? … What kind of [student] understanding of functions would [make you] happy?
>
> B: Well … if they can … given a variety … Given information in a variety of ways, whether it is a table or a graph, or equation, if they can tell me whether it's a function and why, and if they can give me some [examples of those] that's not a function or explain why something is not ... a function, and explain mathematically why it can't be a function. Then I, then I will be satisfied that they've understood ...

For Bud, understanding functions seems to mean being able to determine whether a graph, a table, or an expression is a function and provide examples and non-examples of function. He further indicated that his "students knew the definition of a function, but they couldn't take it and see it in a graph. … They have problems putting the definition to use." He attributed their difficulties to their lack of understanding the concept of ordered pair of numbers and graph.

When asked to consider the need—from the student's viewpoint—of determining whether a given situation is a function, he said:

> …, much of what they are going to do in math relates to the family of functions. …, and then we talk about non-functions, we talk about, I mean, this is the way I've

> always learned it. Here is a function, there is a non-function. The way I learned it, so I am teaching the way I've learned it.

Bud's response suggests that his teaching was content driven rather than student driven. The question of why students would be intrinsically interested in the concept of function is not part of his epistemological or pedagogical consideration. Based on our observations so far, Bud seems to view school mathematics as a fixed set of concepts and procedures that are to be delivered to and remembered by students. These concepts and procedures can be organized systematically into topics and subtopics and can be imparted to students.

In an attempt to advance Bud's knowledge base on the concept of function, we offered him problems whose context can potentially stimulate reasoning in terms of functions. The following problem is an example:

> (I) A pharmacist is to prepare 15 milliliters of special eye drops for a glaucoma patient. The eye-drop solution must have a 2% ingredient, but the pharmacist only has 10% solution and 1% solution in stock. Can the pharmacist use the solutions she has in stock to fill the prescription?
>
> (II) The same pharmacist receives a large number of prescriptions of special eye drops for glaucoma patients. The prescriptions vary in volume but each requires a 2% active ingredient. Help the pharmacist find a convenient way to determine the exact amounts of the 10% solution and 1% solution needed for a given volume of eye drops.

Part II, for example, is likely to help students interpret situations in terms of function. Bud viewed such problem as an application problem appropriate merely for enrichment activities, not to be part of the main curriculum that he is committed to teach. He did use this problem but offered it as the Problem of the Week. He commented that his students had difficulties with the chemistry aspect of the above problem: "I think it wasn't so much the math part that came with the problem, I think more of the problem came from...throwing in chemistry terms into the mix, … the solution and … the terminology." For Bud, the solution of this problem consists of two parts: the process of interpreting the problem statement is not considered mathematics but chemistry. The mathematical part begins when one write the algebraic equations and solves their unknowns.

H: So you have a plan … of how to connect this [the pharmacist problem] to the concept of function.

B: Right, well, umm, I guess I started thinking about more in terms of linear functions instead of functions in general.

H: Oh, linear functions.

B: I don't know if that matters, umm, just cause I originally, as soon as I saw it I just thought two linear equations, that umm, cause I can relate it to, that way I can relate it to slope, I can relate it to y-intercept, I can relate it to solving systems …

Bud viewed this problem as one that can be used to practice linear functions, rather than a situation where one can think in terms of the process conception of function: for any input T (the volume of the prescribed eye-drop solution with 2% active ingredient), one gets the output x (the volume of the 1% solution) from the equation $0.01x + 0.10(T - x) = 0.02T$. Even if Bud did possess the way of thinking of interpreting situations in terms of function, it was not spontaneous for him. As a consequence, he did not attempt to set it as a cognitive objective.

Observations

In his first lesson on the concept of functions, Bud introduced the notions of dependent and independent variables, the definition of function, and the domain and range of a function. He mainly emphasized concept definitions and literal meaning of terms. For example, after discussing the literal meaning of *depend on something*, Bud attempted to relate it to the mathematical meaning of *dependent*. "I depend on the internet [otherwise] I couldn't talk to friends. OK. Just like you guys depend on things, equation has two parts and one part depends on another."

B: Does anybody know what the two parts to an equation are?

S1: The number that makes …

B: Well … No, because we are just talking about the equation. An equation just doesn't have one answer.

S2: Isn't the parts [of both sides of the] equal sign having to be equal to each other?

B: Ummm…

S2: Yes. Say, yes.

B: Kind of yeah but not really what I'm going for.

S3: The independent variable, and the dependent variable (not completely audible)

B: The independent variable and the dependent variable. The independent part and the dependent part.

Bud was particularly focused on his own way of understanding the concept of function that he ignored those of his students. For example, S1 attempted to answer Bud's question "Does anybody know what the two parts to an equation are?" by saying how he understood the meaning of an equation. Rather than trying to build on S1's WoU, Bud chose to reject S1's answer. His style of exchange with students is generally not of a free discussion but of an attempt to deliver his own knowledge.

The following excerpt shows that some students had difficulty with the "uniqueness to the right" property (i.e., for an input value there could be only one output value).

B: … More or less a parabola, a little skewed but that's OK. Is it going to be a function? [S1], why? You are shaking your head.

S1: (inaudible)

B: OK. For each x I have, like say OK, this x right here, [if I'm looking at] this x, how many values of y does it have to match up with?

S2: One.

B: It has one right there. Is there any place on this graph that has more than one y value for the x?

S3: (said something about 2 x values for 1 y value.)

B: Different thing. It's a good question. We will get to that eventually. His question was, what if [we] look at it backwards, I think. What if, you are looking at the y, and say, because this value of the y, you will notice that it has how many x values?

S4: Two.

B: Two. For, being a function, that doesn't matter? Excellent question, we are going to deal with that later. It has to do with inverse functions and things like that. But for now, for functions, all we are looking at, for each x, there is only one y.

Bud did not seem to empathize with students' struggle in understanding why a function must have one y-value for each x-value and not the other way around. Instead of addressing this difficulty, he resorted to a different issue—that of the inverse function—a concept the students had not been exposed to at the time. Student's difficulty with the uniqueness-to-the-right property surfaced again when he discussed whether a line is a function; his students were unable to understand why a horizontal line is a function but a vertical line is not. Bud believed that the concept definition (in the sense of Tall & Vinner, 1981) alone is sufficient for students to overcome their difficulties with the concept of function.

In his lesson on linear function, he discussed the characteristics of linear function, the names and WoU for m and b in $y = mx + b$, the procedure for graphing $y = mx + b$ without plotting points, and the procedure for finding the equation of the line passing through two points whose coordinates are given.

B: What do you think something that is linear is going to look like?

S1: Straight.

S2: Line.

B: Line. So if it is a linear function it could be a straight up and down line then? … It could be a vertical line?

S's: No.

B: Why, why can't I have a vertical line if I want a linear function?

S3: [A vertical line] isn't a function.

B: Right. Vertical lines remember aren't function. When I say … linear functions, I am not talking about vertical lines.

The above excerpt suggests a view that mathematical facts are to be remembered. In the following excerpt, Bud's question "does anybody *remember* how you could do it using slope and y-intercept" suggests that the procedure for sketching is something that one should memorize rather than reconstruct.

B: What, when you graph something, y equals, say $2x$ plus 5, what do you do first?

S1: Plug in the number for x.

S2: Well, I know … (said something about making a table).

B: Well, you could make an in-and-out table. Does anybody remember how you could do it using slope and y-intercept though?

S3: Yes

B: How so?

S3: Get, err, get numbers for x, plug in …(inaudible) …x, you get negative 3.

B: Well, that's making in-and-out table. I want to use; I want to do it without having to make a table. I want to be able to look at the equation and instantly be able to plot points, without having to plug in anything.

Bud chose not to pursue students' suggestions because his goal was to teach the intercept-slope procedure for sketching, a procedure which he considered more efficient. As such, he missed the opportunity to build on students' current knowledge to develop a critical way of thinking, that of appreciation for mathematical efficiency.

CONCLUSION

This preliminary analysis focuses mainly on observations-conversations concerning the concept of function. Bud's ways of understanding and ways of thinking of this concept and the way he taught it give hints as to his knowledge base, which seems to include the following beliefs: (a) mathematics is a fixed set of interrelated concepts and procedures, (b) modeling is not part of algebra, algebra is essentially manipulation of symbols, (c) learning mathematics means essentially remembering what the teacher teaches, (d) content structure, not student need, drive mathematics curricula.

A conceptual framework for teacher's knowledge base would enable us to describe teacher's teaching personalities and the rationale for their teaching actions. We hope that a complete model for Bud's knowledge base would help us explain his preference for teacher-led discussions over lectures, his tendency to disregard students' current ways of understanding, and his choice and sequencing of problems for classroom discussions.

The components of a teacher's knowledge base are inseparable from each other. One's ways of understanding and ways of thinking of mathematical concepts seem to dictate the nature of the other components of knowledge. For example, Bud's way of understanding functions impacted the kind of emphasis he placed on the pharmacist problem. He focused on the procedural aspect of solving the problem rather than the conceptual aspect of modeling the problem situation in terms of functions.

This has implications to curricula for pre-service mathematics teachers and for professional development programs for in-service mathematics teachers. Focusing on one component of teacher knowledge base in isolation from the other two is unlikely to be effective. It is unrealistic, for example, to expect prospective teachers to change their beliefs and conceptions about mathematics they have formed over the years in one or two courses. Integrated curricula, where the three components of knowledge base are addressed in a synergetic manner, can help teachers grapple with the mathematics and at the same time reflect on their own learning, which, in turn, can help them appreciate epistemological and pedagogical issues.

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