AN ASPECT OF MATHEMATICAL UNDERSTANDING: THE NOTION OF "CONNECTED KNOWING"

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Much advice about teaching for understanding implies that teacher should help children to develop connections between aspects of their experience, knowledge, and skills. This paper outlines points from the literature about different types of connections and describes relevant points from four case study teachers said and did.

TYPES OF CONNECTED KNOWING

Few authors claim to have studied teaching mathematics for understanding *per se*. One exception is Knapp et al. (1995), who studied experienced teachers in 15 schools while gathering longitudinal data via standardised tests of mathematical understanding. The features of meaning-oriented instruction identified are:

(a) broadening the range of mathematical content studied to give children a sense of the breadth of mathematics and its applications;
(b) emphasising connections between mathematical ideas;
(c) exploring the mathematics that is embedded in rich, "real life" situations;
(d) encouraging students to find multiple solutions and focusing students' attention on links between the solution processes used; and
(e) creating multiple representations of ideas (e.g., drawings and physical objects).

Every one of these characteristics of teaching mathematics for understanding involves connections, but each implies different types of "connected knowing".

Overall, the research literature available to Australian teachers, many professional development programs, and curriculum advice provided by state and national government bodies suggest that the development of connections in mathematics education is important. Thus how teachers interpret such suggestions, as well as whether and how they facilitate the development of a range of types of connections, are important research questions. My analysis of many relevant articles (see Mousley, 2003) demonstrates that literature on the development of mathematical understanding has a focus on connected knowing, but authors tend not to distinguish between possible meanings of this term. The three most common interpretations were: (a) connections that learners make between new information and existing understandings; (b) relationships between different mathematical ideas and representations; and (c) links that teachers and children make between school concepts and the mathematical aspects of other everyday contexts. There are other possible meanings, such as personal connections that individuals make with educational content (see for example Ocean, 1998), but reference to these is not common.

Over the past five years, I have carried out research into how teachers describe and develop students' mathematical understanding. The remainder of this paper reports on my research into ways that four of the teachers, all from one school in a rural area in Victoria, Australia, attended to the development of "connected knowing".

THE RESEARCH CONTEXT AND METHODS

Case study methods were used. About 4 weeks were spent in each of 4 classrooms. David and Susan were the teachers of the Year 6 (age 1112) classes; Tracey and Frea taught the Year 2 (age 78) classes. Teachers were interviewed several times, and their mathematics lessons were videotaped. The resulting audiotaped and videotaped data were analysed to find examples of what the teachers believed and did in relation to the development of their pupils' mathematical understanding. This paper focuses on a small sub-set of the data. The full report of this (Mousley, 2003) is descriptive, with close reference and electronic links to extensive multimedia appendices.

Making connections between new information and current knowledge

This first interpretation of connected knowing—the most commonly-used—was popularised by Piaget, who portrayed understanding as constructed, developed and organised as a result of cognitive interaction between an sensory experience and existing schema. According to Piaget (1926) and others (e.g., Steffe, Cobb & von Glasersfeld, 1988), sensory input is filtered, arranged, and stored in complex networks of concepts, rules, and strategies. These shape cognition as new experiences are either assimilated into existing schemata or accommodated by a change in their structure. Understanding mathematics involves developing a harmonious network of information that may include images, relationships, errors, hypotheses, anticipations, inferences, inconsistencies, gaps, feelings, rules, and generalisations (O'Brien, 1989).

Many researchers have drawn on this constructivist approach. For example, Hiebert and Lefevre (1986) wrote, "Perhaps 'understanding' is the term used most often to describe the state of knowledge when new mathematical information is connected appropriately to existing knowledge" (p. 4). Similarly, Nickerson (1985) claimed that teaching for understanding involves "the connecting of … relatively newly acquired information to what is known, the weaving of bits of new knowledge into the integrated whole" (p. 234). Pound (1999) proposed that any developmentally appropriate curriculum starts with children's ideas: "the starting point must be the child's current understanding—our efforts must go into helping each child to make the connections which will promote idiosyncratic personal understanding" (p. 51).

Many useful ideas are underpinned by this concept. The notion of "learning trajectories" (e.g., Simon, 1993; Steffe & Ambrosio, 1995), for instance, is based on the idea that new knowledge will further develop what has just been learned. Cobb & McClain (1999) referred to an "instructional sequence [that] takes the form of a conjectured learning trajectory that culminates with the mathematical ideas that constitute our overall instructional intent" (p. 24). Hiebert et al. (1997) used the term "residue", involving the understandings children gain from teaching being used as a basis for further planning, but Sfard and Lincheveski (1994) noted the difficulty of predicting when any sequence of activity will become connected and hence more generally meaningful:

> It often happens that three or four steps in instruction add little to the child's understanding of arithmetic, and, then, with the fifth step, something clicks. The turning points at which a general principle becomes clear to the child cannot be set in advance by the curriculum. (pp. 101–102)

In Australia, advice to mathematics teachers typically includes the point that new ideas need to be connected with what is known:

> We can only take from any situation the parts that … can be linked to some existing ideas we have. … Some learning can readily be accommodated within existing conceptual structures. Other learning requires a relatively simple extension or adjustment of ideas we already have. (Australian Education Council, p. 18)

In my research, videotaped lessons showed that the students' knowledge was drawn on frequently to introduce lessons but very infrequently at other times unless students made spontaneous contributions such as "I remember when we did easier ones last year" (Boy, Grade 2). Early in lessons, there were many teachers' requests for students to recall previous in-school experiences.

> Cast your mind back to last week, … Do you remember how amazed we all were that a baby sperm whale, when it is born, is longer than this room? Do you remember how we estimated metres? Think back. Have a think about how long a metre was. (David)

All 4 teachers encouraged children to "fold back" (Kieren & Pirie, 1991), but when analysing videotapes of such interactions I noted that teachers (probably keen to get the lesson underway) generally chose only higher-performing students to talk about concepts from the previous lesson. Further, all of the teachers commented spontaneously that children do not always remember or think to apply what has been taught in previous lessons.

> They looked at rulers last week. They found things that were about 30cm long. They should have understood 40cm would be longer than a ruler. … With all my years of teaching, it is still surprising … They know something … but they don't remember that when they need it. (Tracey)

Planning for teaching of sequential ideas was also seen as difficult as the result of the range of understandings that children have and their abilities to make connections quickly:

> Some of the children like [Boy, Grade 6] are one step ahead of the class … But some of the others [are not]. I spend time planning the best way to step them through, but then they just rely on me to teach them [sequentially] … the better you structure the learning, the less effort they make. (David)

Making connections between mathematical concepts

The next most commonly-mentioned interpretation of connected knowing, in relation to the development of mathematical understanding, involved students understanding relationships between varied mathematical objects—including links between specific facts, ideas, representations, processes, propositions, etc., as well as links across these categories. For example, Hiebert and Lefevre (1986) wrote that:

> Conceptual knowledge is characterised most clearly as knowledge that is rich in relationships. It can be thought of as a connected web of knowledge, a network in which the linking relationships are as prominent as the discrete pieces of information. Relationships pervade the individual facts and propositions so that all pieces of information are linked to some network. (pp. 3–4)

Hiebert, Carpenter, and colleagues asserted that learning with understanding involves development of an internal network of representations, with the strength of a child's understanding being determined by the number and strength of its connections. They point out that in the development of understanding, communication and reflection are vital:

> Communication works together with reflection to produce new relationships and connections. Students who reflect on what they do and communicate with others about it are in the best position to build useful connections in mathematics. (Hiebert et al., 1997, p. 6)

Skemp's theorising of the concept of "relational understanding", where pedagogy is aimed at the construction of relational cognitive schemata, is relevant here.
A possessor of such schemata can, in principle, produce an unlimited number of plans for getting from any starting point within a schema to any finishing point (Skemp, 1976, p. 23). Byers and Herscovics (1977) distinguished further between "relational understanding", or the ability to deduce particular procedures through a recognition of mathematical relationships, and "formal understanding" that involved the ability to connect mathematical forms of notation with the appropriate ideas and to combine these into sequences of mathematical reasoning. Focusing on teaching for understanding of decimals, Hart (1989) provided some practical examples of the networking of mathematical ideas. In her discussion of the results of a test that focused on place value, she pointed out that:

> It is above all clear that learning of whole numbers … involved internalising a whole chain of relationships and connections, some with the place-structure itself (e.g., 0.9 is equivalent to 0.90) and some linking to other concepts like those of fractions (e.g., the notion of one hundredth and its relationship to one tenth), some visual correspondences and some connecting to applications in the 'real' world. (p. 64)

When asked about ways that they developed children's understanding of mathematics, both of the case study Year 6 teachers volunteered that they made a point of linking operations. When asked for an example, David said "Take place value: it is not only the basis of numerals and the number system but also measurement, decimals, percentages, and so on.” Susan said that she had been to an extended professional development course for primary teachers of mathematics several years previously, and “One thing that was really emphasised was constructivism and ways to help children build knowledge networks where maths ideas are all linked together in meaningful ways”.

However, in the many Grade 6 lessons observed over their 8-week period, emphasis on connections between mathematical forms or operations was rarely explicit. The exception was when percentages were taught and both teachers emphasised the different ways that fractions can be expressed. Generally, though, such understandings needed to be gleaned from school experience. In fact, when asked about whether she emphasises connections between the four processes, Susan acknowledged, “I don’t emphasise them much. Probably only the best of the children are able to make all the necessary connections”.

One of the Year 2 teachers, Frea, thought that making connections between operations had the potential to confuse her pupils. Without prompting, she gave an example.

> Well, I know that multiplication is like adding. [She gave example of repeated addition]. The children might realise that. I am sure some of them do—not many though. No, I think they do not think of it as addition. … I want them to see multiplying as repeated groups, not as repeated addition. That might confuse them. It’s incidental.

I noted several instances where the use of context-based cross-curriculum linkages resulted in cross-mathematics linkages. An example of this was David’s lesson where children made “Flying Machines” from scrap materials, measured the length of flights, averaged the results for each machine, and graphed then wrote about the outcomes. While David saw this as a Science-Mathematics-Language integrated activity, the lesson clearly helped the students to develop connected understandings within mathematics as well as across subject areas.

Making connections to everyday experience

The third common interpretation involves bi-directional connections between the “real world” and school mathematics. In one way, it involves children coming to understand ways that mathematical knowledge and skills can be applied in everyday contexts. Alternatively, children draw on their everyday experiences to understand school mathematics ideas as well as the need for specific mathematical processes.

The aim of facilitating links between mathematics concepts and everyday experiences underpinned the *Connected Mathematics Project*. Here, Ben-Chaim, Fey, Fitzgerald, Benedetto, and Miller (1997) concluded that taking a problem-based approach that draws on children’s knowledge of everyday contexts (see, for example, Zawojewski & Hoover, 1996) could help students to construct effective networks of knowledge, understanding, and skills.

While the idea of basing learning activities on children's out-of-school experiences seems logical, it necessitates "multiplied and more intimate" contacts between teachers and students, and thus "more, rather than less, guidance by others" (Dewey, 1938, p. 8). It is also important to note that some connections that seem obvious to adults prove difficult for children to make. Thomas (2000), for instance, showed that while many children grouped objects in tens to make them easier to count (about 20% in Grade 2 and 60% in Grade 6), only one (Grade 1) child of 132 observed and interviewed was able to explain why it made sense to do that. What most adults take for granted proved to be difficult for the young children to grasp, as it involves not only understanding the structure of the number system and being able to group and count by tens, but also the second form of connected knowing above—understanding of the way that this knowledge and this skill are related.

Again, advice was readily available to available to the case study teachers. For instance, their course advice document, referring to the link between everyday experience and mathematics pedagogy, states that:

> If this connection is not made, students may see school knowledge as different from, and not relevant to, their out-of-school life. … meaningful activities will encourage the learner to explore their understanding of the world and mathematics. At all levels teachers should draw upon students' prior everyday activities to ensure abstract ideas are linked to the familiar. (Victorian Directorate of School Education, 1995, p. 10)

All four of the teachers mentioned this strategy, without prompting and early in the interviews. Susan explained:

> I have always held the philosophy that maths has to be real, that if it can't be applied practically, and made use of in everyday life, its value for most people is questionable. If it is not made real, it is not made understandable. … That's particularly true when I am introducing new ideas. If they can see the use for it, and make the links with real world interests and events, then they will take an interest. But if they can't they just won't make the effort to comprehend.

In the videotaped lessons, however, it was noticeable that there was more time spent in making connections with children's everyday experience when a new topic were introduced than at other times. Such appeals to real contexts were also generally short-lived. Typically, and especially in the teaching and learning of Number, teachers moved quickly from discussion of children's experience to exercises involving abstract examples and work.

All 4 teachers talked about the difficulty of finding appropriate contexts for abstract ideas and operations, but Susan articulated the need for children to be able to refer back to simple understandings of practical contexts as they meet more difficult operations. For example, regarding dividing fractions by inverting the divisor and then multiplying, Susan said:

> When you cut half a cake into quarters you don't need to calculate how many pieces you

have or how big each piece is. But if you do it and discuss the result, at least the kids will have met the idea in a simple form, so they will have some comprehension of not how it works so much but the fact that it does work—and that it can be shown to work.

No overt discussions about how particular mathematics are likely to be applied in everyday contexts in the children's later lives were observed, other than one comment by Tracey: "You have really achieved something, haven't you? You understand it, so you can use it all the time now at home and in school". However, implicit links were made in many of the lessons observed, particularly through the use of word problems, discussions about everyday events (e.g., percentage being used to describe the fat content of foods and in calculation of sale prices) in lesson introductions as well as in sets of application problems used in latter parts of many lessons. Susan described her pragmatics approach: "I think it is not important to have everything useful in their own lives today, so long as they get a sense that maths is sensible and will be useful in all sorts of ways in the future".

CONCLUSION

It is clear that the making of connections is important activity for both teachers and learners in classrooms where teaching is aimed at building mathematical understanding. Articulation of the types of connections that could be made is useful, as categories can be used as a basis for research into teachers' beliefs and actions as well as for pre-service and in-service professional development.

When discussing how they develop children's mathematical understanding, the primary teachers participating in my research referred spontaneously to the three different types of connections that are referred to most commonly in relevant literature and in their formal curriculum documents, but in practice their development of "connected knowing" could have been stronger, more frequent and more consistent. Whether their level of activity in this respect is common in other classrooms, schools and countries is a point for further research, and there is also potential to research how other teachers, texts, curriculum structures, and teacher educators facilitate the development of these three, and other, forms of connections.

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