ON THE C-INTEGRAL

BENEDETTO BONGIORNO

Let $F\colon [a,b] \to \mathbb{R}$ be a differentiable function and let f be its derivative. The problem of recovering F from f is called *problem of primitives*. In 1912, the problem of primitives was solved by A. Denjoy with an integration process (called *totalization*) that includes the Lebesgue integral and the Riemann improper integral. Two years later, a second solution was obtained by O. Perron with a method based on the notions of *major function* and *minor function*. A third solution, based on a generalization of Riemann integral, is due to J. Kurzweil (1957) and R. Henstock (1963).

It is surprising that, nevertheless the three integration processes are completely different, they produce the same integral (i.e. they have the same space of integrable functions and satisfy the same properties).

In 1986, A.M. Bruckner, R.J. Fleissner and J. Foran [9] remarked that the solution provided by Denjoy, Perron, Kurzweil and Henstock possesses a generality which is not needed for this purpose. In fact the function

$$F(x) = \begin{cases} x \sin \frac{1}{x^2} & , \ 0 < x \leq 1 \\ 0 & , \ x = 0 \end{cases} \tag{1}$$

is a primitive for the Denjoy-Perron-Kurzweil-Henstock integral (more precisely, F is a primitive for the Riemann improper integral), but it is neither a Lebesgue primitive, neither a differentiable function, nor a sum of a Lebesgue primitive and a differentiable function (see [9] for details).

The question of providing a minimal constructive integration process which includes the Lebesgue integral and also integrates the derivatives of differentiable functions was solved by the following Riemann-type integral:

Supported by MURST of Italy.

Definition 1. Given a function $f\colon [a,b] \to \mathbf{R}$ we say that f is C-integrable on $[a,b]$ if there exists a constant A such that for each $\varepsilon > 0$ there is a gauge δ on $[a,b]$ with

$$\left| \sum_{i=1}^{p} f(x_i)|I_i| - A \right| < \varepsilon \ , \tag{2}$$

for each δ-fine McShane-partition $\{(I_1, x_1), \dots, (I_p, x_p)\}$ of [a,b] satisfying the condition

$$\sum_{i=1}^{p} \operatorname{dist}(x_i, I_i) < 1/\varepsilon. \tag{3}$$

Here by "gauge" on $[a,b]$ we mean a positive function δ defined on $[a,b]$, and by "δ-fine McShane-partition of $[a,b]$" we mean a collection $\{(I_1, x_1), \dots, (I_p, x_p)\}$ of pairwise nonoverlapping intervals $I_i \subset [a,b]$ and points $x_i \in [a,b]$ such that $I_i \subset (x_i - \delta(x_i), x_i + \delta(x_i))$ and $\sum_{i=1}^{p} |I_i| = b - a$.

The number A is called the C-integral of f on $[a,b]$, and we set $A = (C)\int_a^b f$.

The mentioned minimality of the C-integral as a constructive integration process which includes the Lebesgue integral and also integrates the derivatives of differentiable functions follows by the following theorem.

Theorem 2. ([5, Main Theorem]) *A function $f\colon [a,b] \to \mathbf{R}$ is C-integrable on $[a,b]$ if and only if there exists a derivative h such that $f - h$ is Lebesgue integrable.*

1. Relations between the C-integral and the Lebesgue integral, the Henstock-Kurzweil integral, and the Riemann-improper integral

It is well known that the Lebesgue integral is equivalent to the McShane integral ([16], [17]); consequently, by Definition 1 it follows that each Lebesgue integrable function is C-integrable with the same value of the integral.

Notice also that if f is C-integrable on $[a,b]$, then f is Henstock-Kurzweil integrable on $[a,b]$ with the same value of the integral. In fact, condition (2) holds for each collection $\{(I_1, x_1), \dots, (I_p, x_p)\}$ of pairwise nonoverlapping intervals $I_i \subset [a, b]$ and points $x_i \in [a, b]$ such that $I_i \subset (x_i - \delta(x_i), x_i + \delta(x_i))$ and $x_i \in I_i$, $i = 1, 2, \dots, p$.

Now we prove that both inclusions are proper.

Proposition 3. *The Lebesgue integral is properly contained into the C-integral; i.e. there is a C-integrable function f such that f is not Lebesgue integrable.*

Proof. Let us consider the function f on $[0, 1]$ given by

$$f(x) = 2x \sin\frac{1}{x^2} - \frac{2}{x}\cos\frac{1}{x^2}, \text{ with } f(0) = 0.$$

A primitive of f is the function F given by $F(x) = x^2 \sin(1/x^2)$, $F(0) = 0$. It is easy to check that F is not absolutely continuous on $[0, 1]$; hence f is not Lebesgue integrable on $[0, 1]$. Now remark that F is differentiable everywhere in $[0, 1]$ with derivative $F'(x) = f(x)$, $x \in [0, 1]$. So f is C-integrable, by the following proposition.

Proposition 4. *Each derivative is C-integrable.*

Proof. Let F be an everywhere differentiable function on $[a, b]$, and let $f(x) = F'(x)$ for each $x \in [a, b]$. Given $0 < \varepsilon < 1/(b - a)$ and $x \in [a, b]$, by definition of derivative, there exists $\delta(x) > 0$ such that

$$\left| \frac{F(y) - F(x)}{y - x} - f(x) \right| < \frac{\varepsilon^2}{4}, \tag{4}$$

for each $y \in [a, b]$ with $|y - x| < \delta(x)$.
Given an interval $I = (\alpha, \beta)$ we set $F(I) = F(\beta) - F(\alpha)$ and $|I| = \beta - \alpha$. If I is a subinterval of $(x - \delta(x), x + \delta(x))$, then by

(4) we have

$$
\begin{aligned}
&| F(I) - f(x)|I|| \\
&\quad \leq |F(\beta) - F(x) - f(x)(\beta - x)| \\
&\qquad + |F(\alpha) - F(x) - f(x)(\alpha - x)| \\
&\quad < \frac{\varepsilon^2}{4}|\beta - x| + \frac{\varepsilon^2}{4}|\alpha - x| \\
&\quad \leq \frac{\varepsilon^2}{2}\ (\mathrm{dist}(x, I) + |I|).
\end{aligned}
\tag{5}
$$

Therefore, if $\{(I_1, x_1), \cdots, (I_p, x_p)\}$ is a δ-fine McShane-partition of $[a, b]$ satisfying the condition $\sum_{i=1}^{p} \mathrm{dist}(x_i, I_i) < 1/\varepsilon$, then by (5) we get

$$
\begin{aligned}
&\left| \sum_{i=1}^{p} f(x_i)|I_i| - (F(b) - F(a)) \right| \\
&\quad \leq \sum_{i=1}^{p} |f(x_i)|I_i| - F(I_i)| < \frac{\varepsilon^2}{2} \sum_{i=1}^{p} (\mathrm{dist}(x_i,\ I_i) + |I_i|) \\
&\quad < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}
$$

Thus f is C-integrable on $[a, b]$. □

Proposition 5. *The C-integral is properly contained into the Henstock-Kurzweil integral; i.e. there is a Henstock-Kurzweil integrable function f such that f is not C-integrable.*

Proof. Let us consider the function (1) and define $f = F'$ in $(0, 1]$ and $f(0) = 0$. It is easy to see that f is Riemann improper integrable on $[0, 1]$, hence f is Henstock-Kurzweil integrable. To prove that f is not C-integrable we need the following extension of classical Henstock's lemma:

Lemma 6. *If a function $f\colon [a, b] \to \mathbb{R}$ is C-integrable on $[a, b]$, then given $\varepsilon > 0$ there exists a gauge δ on $[a, b]$ so that*

$$
\sum_{i=1}^{p} \left| f(x_i)|I_i| - (C) \int_{I_i} f \right| < \varepsilon\ , \tag{6}
$$

for each δ-fine partial McShane-partition $\{(I_1, x_1), \dots, (I_p, x_p)\}$ *in* $[a, b]$ *with* $\sum_{i=1}^p \mathrm{dist}(x_i, I_i) < 1/\varepsilon$.

Here by "δ-fine partial McShane-partition $\{(I_1, x_1), \dots, (I_p, x_p)\}$ in $[a, b]$" we mean a collection $\{(I_1, x_1), \dots, (I_p, x_p)\}$ of pairwise nonoverlapping intervals $I_i \subset [a, b]$ and points $x_i \in [a, b]$ such that $I_i \subset (x_i - \delta(x_i), x_i + \delta(x_i))$ and $\sum_{i=1}^p |I_i| < b - a$.

The proof of this lemma follows easily by a simple adaptation of standard technics (see [13, Lemma 9.11]).

Assume, by contradiction, that f is C-integrable on $[0, 1]$. Given $\varepsilon > 0$, take a gauge δ according with Lemma 6, and let $a_h = (\pi + 2h\pi)^{-1/2}$, $b_h = (\pi/2 + 2h\pi)^{-1/2}$. It is easy to see that $\sum_h a_h = \sum_h b_h = \infty$, and that the intervals (a_h, b_h), $h = 1, 2, \dots$, are pairwise disjoint. Moreover, by (1) we have $F(a_h) = 0$, $F(b_h) = b_h$, for each $h \in \mathbb{N}$. Take natural numbers n, p such that

$$(a_{n+i}, b_{n+i}) \subset (0, \delta(0)), \ i = 1, \cdots, p, \quad \text{and} \ \ \varepsilon < \sum_1^p a_{n+i} < \frac{1}{\varepsilon}.$$

Hence $\sum_1^p b_{n+i} > \sum_1^p a_{n+i} > \varepsilon$. Now define $I_1 = (a_{n+1}, b_{n+1}), \cdots$, $I_p = (a_{n+p}, b_{n+p})$. Then $\{(I_1, 0), \dots, (I_p, 0)\}$ is a δ-fine partial McShane-partition in $[0, 1]$, and $\sum_{i=1}^p \mathrm{dist}(0, I_i) = \sum_{i=1}^p a_i < 1/\varepsilon$. Moreover

$$\sum_{i=1}^p \left| f(0)|I_i| - (C)\int_{I_i} f \right| = \sum_{i=1}^p |F(b_{n+i}) - F(a_{n+i})| = \sum_{i=1}^p b_{n+i} > \varepsilon,$$

in contradiction with (6). Thus f is not C-integrable on $[0, 1]$. $\square$

Remark, moreover, that the function (1) is Riemann improper integrable on $[0, 1]$. Hence next proposition is also proved.

Proposition 7. *The C-integral doesn't contain the Riemann improper integral.*

2. The variational measure $V_C F$

A first descriptive characterization of the C-primitives was obtained by A.M. Bruckner et al. [9]. They proved that a function F is a C-primitive if and only if F is the limit in variation of a sequence of absolutely continuous functions.

Note that, in general, F is not a function of bounded variation; however, its associate variational measure is absolutely continuous with respect to the Lebesgue measure. This follows from the observation that a C-primitive is also a Henstock-Kurzweil primitive and by [6, Theorem 3].

In fact the last mentioned theorem gives a characterization of Henstock-Kurzweil primitives. The idea of considering appropriate variational measures to characterize the primitives of some integral has been also used in [4, 10, 11] for many multidimensional integrals and in [7] for the Henstock-dyadic integral and for the Henstock-symmetric integral.

Given a function F on $[a, b]$, a gauge δ on $[a, b]$, a subset E of $[a, b]$, and a positive ε, we denote by $\mathrm{V}_\varepsilon(F, \delta, E)$ the supremum of all sums $\sum_i |F(I_i)|$ where $\{(I_1, x_1), \ldots, (I_p, x_p)\}$ runs into the class of δ-fine partial McShane-partitions of $[a, b]$ with $x_i \in E$, $i = 1, 2, \cdots p$, and $\sum_{i=1}^{p} \mathrm{dist}(x_i, I_i) < 1/\varepsilon$.

It is clear that

$$\mathrm{V}_{\varepsilon_1}(F, \delta, E) \leq \mathrm{V}_{\varepsilon_2}(F, \delta, E), \quad \text{for } \varepsilon_1 > \varepsilon_2. \tag{7}$$

Then we define

$$V_C F(E) = \sup\nolimits_\varepsilon \inf\nolimits_\delta \mathrm{V}_\varepsilon(F, \delta, E).$$

By the same argument used in [20] for proving Theorem 3.7 and Theorem 3.15, we can show that the extended real-valued set function $V_C F$ is a *Borel regular measure* in $[a, b]$.

Theorem 8. [2, Theorem 4.1] *F is an indefinite C-integral if and only if the variational measure $V_C F$ is absolutely continuous with respect to the Lebesgue measure.*

This characterization is used to prove the following theorem.

Theorem 9. [2, Theorem 4.2] *Every BV function is a multiplier for the C-integral; i.e. if g is a BV function on $[a, b]$ and f is C-integrable on $[a, b]$, then fg is also C-integrable on $[a, b]$.*

Here by BV function we mean a function $g\colon [a, b] \to \mathbb{R}$ such that there exists a function of bounded variation $\tilde{g}\colon [a, b] \to \mathbb{R}$ with $g = \tilde{g}$ almost everywhere in $[a, b]$.

It is known that the product of two derivatives may be not a derivative. A simple consequence of Theorem 9 show that

Theorem 10. [2, Theorem 4.3] *The product between a derivative and a BV function is a derivative modulo a Lebesgue integrable function of arbitrarily small L^1-norm; i.e. if f is a derivative and g is a BV function, then there is a sequence $\{\phi_n\}$ of derivatives such that $\|fg - \phi_n\|_1 \to 0$.*

3. ACG_C Functions

Definition 11. A function $F\colon [a,b] \to \mathbb{R}$ is said to be C-absolutely continuous on a set $E \subset [a,b]$ (br. $AC_C(E)$) whenever for each $\varepsilon > 0$ there exist a constant $\eta > 0$ and a gauge δ on E such that

$$\sum_i |F(I_i)| < \varepsilon \tag{8}$$

for each δ-fine partial McShane-partition $\{(I_1, x_1), \cdots, (I_p, x_p)\}$ in $[a,b]$ satisfying the following conditions:

(α_1) $x_i \in E$, $i = 1, 2, \ldots, p$;
(α_2) $\sum_{i=1}^{p} \operatorname{dist}(x_i, I_i) < 1/\varepsilon$;
(α_3) $\sum_i |I_i| < \eta$.

Definition 12. A function $F\colon [a,b] \to \mathbb{R}$ is said to be C-generalized absolutely continuous on $[a,b]$ (br. $ACG_C[a,b]$) whenever

(β) there exist measurable sets $E_1, E_2, \ldots$ such that $[a,b] = \bigcup_n E_n$ and F is $AC_C(E_n)$, $n = 1, 2, \ldots$.

According with R.A. Gordon [12] a function $F\colon [a,b] \to \mathbb{R}$ is said to be $AC_\delta(E)$ whenever condition (8) is satisfied for each δ-fine partial McShane-partition $\{(I_1, x_1), \cdots, (I_p, x_p)\}$ satisfying conditions (α_1), (α_3), and

($\alpha_2{}'$) $\sum_{i=1}^{p} \operatorname{dist}(x_i, I_i) = 0$.

Moreover F is said to be $ACG_\delta[a,b]$ whenever

(β') there exist measurable sets $E_1, E_2, \ldots$ such that $[a,b] = \bigcup_n E_n$ and F is $AC_\delta(E_n)$, $n = 1, 2, \ldots$.

Therefore each AC_C-function is AC_δ, and each ACG_C-function is ACG_δ.

Lemma 13. *If F is $ACG_C[a,b]$, and $E \subset [a,b]$ with $|E| = 0$, then for each $\varepsilon > 0$ there is a gauge δ on E such that*

$$\sum_{i=1}^{p} |F(I_i)| < \varepsilon,$$

for each δ-fine partial McShane-partition $\{(I_1, x_1), \cdots, (I_p, x_p)\}$ in $[a,b]$ satisfying the following conditions

(α_1) $x_i \in E$, $i = 1, 2, \ldots, p$;
(α_2) $\sum_{i=1}^{p} \mathrm{dist}(x_i, I_i) < 1/\varepsilon$.

Theorem 14. *A function F is $ACG_C[a,b]$ if and only if there is a C-integrable on $[a,b]$ function f such that*

$$F(x) - F(a) = (C)\int_a^x f(t)\,dt, \quad \text{for each} \quad x \in [a,b]. \tag{9}$$

Proof. Assume that F is $ACG_C[a,b]$. Since F is $ACG_\delta[a,b]$, by [12, Theorem 6] and by [19, Chapter VII, Theorem 7.2], F is differentiable almost everywhere in $[a,b]$. Let E be the set of points $x \in [a,b]$ such that F is not differentiable at x. Then $|E| = 0$. So, by Lemma 13, given $0 < \varepsilon \leq 1/(b-a)$ there is a gauge τ on $[a,b]$ such that

$$\sum_{i=1}^{p} |F(I_i)| < \frac{\varepsilon}{4},$$

for each τ-fine partial McShane-partition $\{(J_1, x_1), \cdots, (J_p, x_p)\}$ in $[a,b]$ with $\sum_{i=1}^{p} \mathrm{dist}(x_i, I_i) < 1/\varepsilon$ and $x_i \in E$, $i = 1, 2, \ldots$.

If F is differentiable at y then, by (5) we can find a positive constant $\gamma(y)$ such that

$$|\, F(I) - F'(y)|I|\, | < \frac{\varepsilon^2}{2}(\mathrm{dist}(y, I) + |I|),$$

for each interval $I \subset (y - \gamma(y), y + \gamma(y))$.
Define

$$\delta(y) = \begin{cases} \tau(y) & \text{if } y \in E, \\ \gamma(y) & \text{if } y \notin E, \end{cases}$$

and

$$f(y) = \begin{cases} 0 & \text{if } y \in E, \\ F'(y) & \text{if } y \notin E. \end{cases}$$

Given $x \in [a,b]$ let $\{(I_1,x_1),\cdots,(I_p,x_p)\}$ be a δ-fine McShane-partition of $[a,x]$ such that $\sum_{i=1}^p \operatorname{dist}(x_i,I_i) < 1/\varepsilon$. Then we have

$$\begin{aligned}
&\left|\sum_{i=1}^p f(x_i)|I_i| - (F(x)-F(a))\right| \\
&\qquad \le \sum_{i=1}^p |f(x_i)|I_i| - F(I_i)| \\
&\qquad < \sum_{x_i\in E} |F(I_i)| + \sum_{x_i\notin E} |F'(x_i)|I_i| - F(I_i)| \\
&\qquad < \frac{\varepsilon}{2} + \sum_{x_i\notin E} \frac{\varepsilon^2}{4}(\operatorname{dist}(x_i,I_i)+|I_i|) \\
&\qquad < \frac{\varepsilon}{2} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4}\varepsilon(b-a) \le \varepsilon.
\end{aligned}$$

Thus f is C-integrable on $[a,x]$ and (9) holds. Taking $x=b$ we see that f is C-integrable on $[a,b]$.

Now assume that f is C-integrable on $[a,b]$ and let F be the C-primitive of f. For each natural number n we define $E_n = \{x \in [a,b] : |f(x)| \le n\}$. Then $[a,b] = \bigcup_n E_n$. To complete the proof it is enough to show that F is AC_C on E_n, for each n. By Lemma 6, given $\varepsilon > 0$ there is a gauge δ on $[a,b]$ such that

$$\sum_{i=1}^p |f(x_i)|I_i| - F(I_i)| < \frac{\varepsilon}{2},$$

for each δ-fine partial McShane-partition $\{(I_1,x_1),\cdots,(I_p,x_p)\}$ in $[a,b]$ with $\sum_{i=1}^p \operatorname{dist}(x_i,I_i) < 1/\varepsilon$. Assume that $x_i \in E_n$, $i = 1,2,\ldots$, and $\sum_i |I_i| < \varepsilon/2n$. Then

$$\begin{aligned}
&\sum_{i=1}^p |F(I_i)| \\
&\qquad \le \sum_{i=1}^p |f(x_i)|I_i| - F(I_i)| + \sum_{i=1}^p |f(x_i)|\cdot|I_i| \\
&\qquad < \frac{\varepsilon}{2} + n\sum_i |I_i| < \varepsilon.
\end{aligned}$$

So F is $AC_C(E_n)$, and the proof is complete. □

4. Convergence Theorems.

Let f be Lebesgue (resp. Henstock-Kurzweil) integrable on $[a,b]$. We denote by $(L)\int_a^b f$ (resp. $(HK)\int_a^b f$) the Lebesgue (resp. Henstock-Kurzweil) integral of f on $[a,b]$.

Monotone convergence theorem. *Let $f_1 \leq f_2 \leq \cdots \leq f_n \cdots$ be a sequence of C-integrable functions on $[a,b]$ such that the limit $\lim_n (C)\int_a^b f_n$ exists finite. Then the function $f(x) = \lim_n f_n(x)$ is C-integrable on $[a,b]$ and we have*

$$(C)\int_a^b f(x)\,dx = \lim_{n\to\infty} (C)\int_a^b f_n(x)\,dx.$$

Proof. Since each C-integrable function is Henstock-Kurzweil integrable, by [18, Corollary 6.3.5] each nonnegative C-integrable function is Lebesgue integrable, and by [18, Theorem 6.3.3] each C-integrable function is Lebesgue measurable. Then we can apply the Lebesgue monotone convergence theorem to the following sequence

$$0 \leq f_2 - f_1 \leq f_3 - f_1 \leq \cdots f_n - f_1 \leq \cdots$$

Hence

$$(L)\int_a^b \{f(x) - f_1(x)\}\,dx = \lim_{n\to\infty} (L)\int_a^b \{f_n(x) - f_1(x)\}\,dx. \tag{10}$$

Now for each $n \in \mathbb{N}$ we have

$$\begin{aligned}
&(L)\int_a^b \{f_n(x) - f_1(x)\}\,dx \\
&\quad = (C)\int_a^b \{f_n(x) - f_1(x)\}\,dx \\
&\quad = (C)\int_a^b f_n(x)\,dx - (C)\int_a^b f_1(x)\,dx.
\end{aligned}$$

Therefore, since $\lim_n (C)\int_a^b f_n$ exists finite, we can use (10) to infer that $f - f_1$ is Lebesgue integrable; hence $f = (f - f_1) + f_1$ is C-integrable. Moreover

$$
\begin{aligned}
&(C)\int_a^b f(x)\,dx \\
&\quad = (C)\int_a^b \{f(x) - f_1(x)\}\,dx + (C)\int_a^b f_1(x)\,dx \\
&\quad = \lim_{n\to\infty} (C)\int_a^b f_n(x)\,dx - (C)\int_a^b f_1(x)\,dx + (C)\int_a^b f_1(x)\,dx \\
&\quad = \lim_{n\to\infty} (C)\int_a^b f_n(x)\,dx.
\end{aligned}
$$

□

Dominated convergence theorem. *Let $f_1, f_2, \cdots, f_n \cdots$ be a sequence of measurable functions such that*

(i) $f_n(x) \to f(x)$ *almost everywhere in* $[a, b]$;
(ii) $g(x) \le f_n(x) \le h(x)$, *almost everywhere in* $[a, b]$, *with* g *and* h C*-integrable on* $[a, b]$;

then f *is* C*-integrable on* $[a, b]$ *and*

$$(C)\int_a^b f(x)\,dx = \lim_{n\to\infty} (C)\int_a^b f_n(x)\,dx.$$

Proof. By (ii) we have $0 \le f_n - g \le h - g$, almost everywhere in $[a, b]$. Moreover $h-g$ is Lebesgue integrable, since non negative and C-integrable. Then, by Lebesgue dominated convergence theorem, $f - g$ is Lebesgue integrable on $[a, b]$ with

$$(L)\int_a^b \{f(x) - g(x)\}\,dx = \lim_{n\to\infty} (L)\int_a^b \{f_n(x) - g(x)\}\,dx.$$

Therefore, by $f = (f - g) + g$ we infer that f is C-integrable on $[a, b]$ with

$$\begin{aligned}(C)\int_a^b f(x)\,dx &= (C)\int_a^b \{f(x) - g(x)\}\,dx + (C)\int_a^b g(x)\,dx \\ &= (L)\int_a^b \{f(x) - g(x)\}\,dx + (C)\int_a^b g(x)\,dx \\ &= \lim_{n\to\infty} (C)\int_a^b f_n(x)\,dx - (C)\int_a^b g(x)\,dx + (C)\int_a^b g(x)\,dx \\ &= \lim_{n\to\infty} (C)\int_a^b f_n(x)\,dx.\end{aligned}$$

□

Definition 15. Let E be a subset of $[a, b]$ and let $\{F_n\}$ be a sequence of real valued functions defined on $[a, b]$. It is said that $\{F_n\}$ is *uniformly* $AC_C(E)$ if for each $\varepsilon > 0$ there exist a constant $\eta > 0$ and a gauge δ such that

$$\sup_n \sum_i |F_n(I_i)| < \varepsilon,$$

for each δ-fine partial McShane-partition $\{(I_1, x_1), \cdots, (I_p, x_p)\}$ in $[a, b]$, satisfying conditions (α_1), (α_2), (α_3).

Definition 16. A sequence $\{F_n\}$ is said to be *uniformly* $ACG_C[a, b]$ if there exist measurable sets $E_1, E_2, \ldots$ such that $[a, b] = \bigcup_k E_k$ and $\{F_n\}$ is uniformly $AC_C(E_k)$, $k = 1, 2, \ldots$.

Analogously we define the notions of sequence uniformly $AC_\delta(E)$ and uniformly $ACG_\delta[a, b]$. It is easy to see that if a sequence $\{F_n\}$ is uniformly $ACG_C[a, b]$ then $\{F_n\}$ is uniformly $ACG_\delta[a, b]$.

Controlled convergence theorem. *Let $\{f_n\}$ be a sequence of C-integrable on $[a, b]$ functions such that*

(*i*) *$f_n(x) \to f(x)$ almost everywhere in $[a, b]$;*
(*ii*) *the sequence $F_n(x) = (C)\int_a^x f_n(t)\,dt$ is uniformly $ACG_C[a, b]$.*
Then f is C-integrable on $[a, b]$ and

$$(C)\int_a^b f(x)\,dx = \lim_{n\to\infty}(C)\int_a^b f_n(x)\,dx. \tag{11}$$

Proof. Since each C-integrable function is Henstock-Kurzweil integrable and since each uniformly ACG_C sequence is uniformly ACG_δ, then by [3, Theorem 4.1] f is Henstock-Kurzweil integrable on $[a,b]$ and

$$(HK)\int_a^x f(t)\,dt = \lim_{n\to\infty}(HK)\int_a^x f_n(t)\,dt, \tag{12}$$

for each $x \in [a,b]$. Now, by (ii), there exist a sequence of measurable sets $\{E_h\}$ such that

- $[a,b] = \bigcup_h E_h$,
- for each $\varepsilon > 0$ and each h there exist a constant $\eta_h > 0$ and a gauge δ_h such that

$$\sup_n \sum_i |F_n(I_i)| < \varepsilon, \tag{13}$$

for each δ_h-fine partial McShane-partition $\{(I_1,x_1),\cdots,(I_p,x_p)\}$ in $[a,b]$, satisfying conditions (α_1), (α_2), (α_3), with E_h for E.

Set $F(x) = (HK)\int_a^x f(t)\,dt$. Since

$$F_n(x) = (C)\int_a^x f_n(t)\,dt = (HK)\int_a^x f_n(t)\,dt,$$

by (12) we have $F(x) = \lim_n F_n(x)$, for each $x \in [a,b]$. Consequently, by (13), for each $h \in \mathbb{N}$ and for each δ_h-fine partial McShane-partition $\{(I_1,x_1),\cdots,(I_p,x_p)\}$ in $[a,b]$ satisfying conditions (α_1), (α_2), (α_3), with E_h for E, we have

$$\sum_{i=1}^p |F(I_i)| = \lim_{n\to\infty}\sum_{i=1}^p |F_n(I_i)| \le \varepsilon.$$

Thus F is $ACG_C[a,b]$ and by Theorem 14, $F(x) = (C)\int_a^x f(t)\,dt$ for each $x \in [a,b]$. In conclusion (11) follows easily by (12). □

References

[1] B. Bongiorno, Un nuovo integrale per il problema delle primitive, *Le Matematiche*, **51** (1996), No. 2, 299–313.

[2] B. Bongiorno, On the minimal solution of the problem of primitives, *J. Math. Anal. Appl.* **251** (2000), 479–486.

[3] B. Bongiorno, and L. Di Piazza, Convergence theorems for generalized Riemann Stieltjes integrals, *Real Analysis Exchange*, **17** (1991-92), 339–362.

[4] B. Bongiorno, L. Di Piazza, and D. Preiss, Infinite variation and derivatives in $\mathbf{R}^n$, *J. Math. Anal. Appl.*, **224** (1998), No. 1, 22–33.

[5] B. Bongiorno, L. Di Piazza, and D. Preiss, A constructive minimal integral which includes Lebesgue integrable functions and derivatives, *J. London Math. Soc.* (2) **62** (2000), no. 1, 117–126.

[6] B. Bongiorno, L. Di Piazza, and V. Skvortsov, A new full descriptive characterization of Denjoy-Perron integral, *Real Anal. Exchange* **21** (1995/96), No. 2, 656–663.

[7] B. Bongiorno, L. Di Piazza, and V. Skvortsov, On variational measures related to some bases, *J. Math. Anal. Appl.* **250** (2000), no. 2, 533–547.

[8] B. Bongiorno and V. Skvortsov, Multipliers for some generalized Riemann integrals in the real line, *Real Anal. Exchange*, **20** (1994-95), 212-218.

[9] A.M. Bruckner, R.J. Fleissner, and J. Foran, The minimal integral which includes Lebesgue integrable functions and derivatives, *Coll. Math.*, **50** (2) (1986), 289–293.

[10] Z. Buczolich and W. Pfeffer, On absolute continuity, *J. Math. Anal. Appl.* **222** (1998), No. 1, 64–78.

[11] L. Di Piazza, Variational measures in the theory of the integration in $\mathbb{R}^m$, *Czechoslovach Math. J.* **51**(126) (2001), no. 1, 95–110.

[12] R.A. Gordon, A descriptive characterization of the generalized Riemann integral, *Real Anal. Exchange*, **15** (1989-90), 397–400.

[13] R.A. Gordon, *The integrals of Lebesgue, Denjoy, Perron, and Henstock*, Studies in Mathematics **4**, Amer. Math. Soc., Providence, 1994.

[14] R. Henstock, *The general theory of integration*, Clarendon Press, Oxford, 1991.

[15] J. Kurzweil, Generalized ordinary differential equations and continuous dependence on a parameter, *Czechoslovac. Math. J.*, **7** (1957), 418-446.

[16] E.J. McShane, A unified theory of integration, *Amer. Math. Monthly*, **80** (1973), 349–359.

[17] E.J. McShane, *Unified integration*, Academic Press, New York, 1983.

[18] W.F. Pfeffer, *The Riemann Approach to Integration*, Cambridge University Press, Cambridge, 1993.

[19] S. Saks, *Theory of the integral*, Dover, New York, 1964.

[20] B.S. Thomson, *Derivatives of Interval Functions*, Mem. Amer. Math. Soc., **452**, Providence, 1991.

Department of Mathematics, University of Palerme,, Via Archirafi 34, 90123 Palermo (Italy)
E-mail address: bb@math.unipa.it