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A FORMULAE FOR THE SPECTRAL PROJECTIONS OF TIME OPERATOR

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Abstract. In this paper, we study the one-level Friedrichs model by using the quantum time super-operator that predicts the excited state decay inside the continuum. Its survival probability decays exponentially in time.

1. Introduction

In this paper we shall study the concept of survival probability of an unstable quantum system introduced in [6] and we shall test it in the Friedrichs model [7]. The survival probability should be a monotonically decreasing time function and this property could not exists in the framework of the usual Weisskopf-Wigner approach [1, 2, 8, 11]. It could only be properly treated if it is defined through an observable T (time operator) whose eigenprojections provide the probability distribution of the time of decay. The equation defining the **time operator** T is

$$U_{-t}TU_t = T + tI \tag{1}$$

where U_t is the unitary group of states evolution. It is known that such an operator cannot exist when the evolution is governed by the Schrödinger equation, since the Hamiltonian has a bounded spectrum from below, and this contradicts the equation

$$[H, T] = \mathrm{i}I \tag{2}$$

in the Hilbert space of pure states $\mathcal{H}$. However, the time operator T can exist under some conditions, for mixed states. They can be embedded [3, 6, 12] in the "Liouville space", denoted $\mathcal{L}$, that is the space of Hilbert-Schmidt operators ρ on $\mathcal{H}$ such that $\mathrm{Tr}(\rho^*\rho) < \infty$, equipped with the scalar product $<\rho, \rho'>= \mathrm{Tr}(\rho^*\rho')$.

The time evolution of these operators is given by the Liouville von-Neumann group of operators

$$U_t\rho = \mathrm{e}^{-\mathrm{i}tH}\rho\mathrm{e}^{\mathrm{i}tH}. \tag{3}$$

The infinitesimal self-adjoint generator of this group is the Liouville von-Neumann operator L given by

$$L\rho = H\rho - \rho H. \tag{4}$$

That is, $U_t = \mathrm{e}^{-itlL}$. The states of a quantum system are defined by normalized elements $\rho \in \mathcal{L}$ with respect to the scalar product, the expectation of T in the state ρ is given by

$$\langle T\rangle_\rho = \langle \rho, T\rho\rangle \tag{5}$$

and the "uncertainty" of the observable T as its fluctuation in the state ρ

$$(\Delta T)_\rho = \sqrt{\langle T^2\rangle_\rho - (\langle T\rangle_\rho)^2}. \tag{6}$$

Let $\mathcal{P}_\tau$ denote the family of spectral projection operators of T defined by

$$T = \int_{\mathbb{R}} \tau \mathrm{d}\mathcal{P}_\tau. \tag{7}$$

It is shown that [6] the unstable states are those states verifying $\rho = \mathcal{P}_0\rho$. Hence, in the Liouville space, given any initial state ρ, its survival probability in the unstable space is given by

$$p_\rho(t) = \|\mathcal{P}_0\mathrm{e}^{-itL}\rho\|^2 = \|U_{-t}\mathcal{P}_0U_t\rho\|^2 = \|\mathcal{P}_{-t}\rho\|^2. \tag{8}$$

Then, the survival probability is monotonically decreasing to 0 as $t \to \infty$. This survival probability and the probability of finding the system to decay sometime in the interval $I =]0, t]$, $\mathcal{K}_\rho(t)$ are related by

$$\|\mathcal{K}_\rho(t)\|^2 = 1 - p_\rho(t). \tag{9}$$

2. Spectral Projections of Time Operator

The expression of time operator is given in a spectral representation of H. As shown in [6], H should have an unbounded absolutely continuous spectrum. In the simplest case, we shall suppose that H is represented as the multiplication operator on $\mathcal{H} = L^2(\mathbb{R}^+)$

$$H\psi(\lambda) = \lambda\psi(\lambda) \tag{10}$$

the Hilbert-Schmidt operators on $L^2(\mathbb{R}^+)$ correspond to the square-integrable functions $\rho(\lambda, \lambda') \in L^2(\mathbb{R}^+ \times \mathbb{R}^+)$ and the Liouville-von Neumann operator L is given by

$$L\rho(\lambda, \lambda') = (\lambda - \lambda')\rho(\lambda, \lambda'). \tag{11}$$

Then we obtain a spectral representation of L via the change of variables

$$\nu = \lambda - \lambda' \tag{12}$$

and

$$E = \min(\lambda, \lambda'). \tag{13}$$

This gives a spectral representation of L

$$L\rho(\nu, E) = \nu\rho(\nu, E) \tag{14}$$

where $L\rho(\nu, E) \in L^2(\mathbb{R} \times \mathbb{R}^+)$. In this representation $T\rho(\nu, E) = \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\nu}\rho(\nu, E)$ so that the spectral representation of T is obtained by the inverse Fourier transform

$$\hat{\rho}(\tau, E) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}\tau\nu} \rho(\nu, E) d\nu = (\mathcal{F}^* \rho)(\tau, E) \tag{15}$$

and

$$T\hat{\rho}(\tau, E) = \tau\hat{\rho}(\tau, E). \tag{16}$$

The spectral projection operators $\mathcal{P}_s$ of T are given in the (τ, E)-representation by

$$\mathcal{P}_s \hat{\rho}(\tau, E) = \chi_{]-\infty,s]}(\tau)\hat{\rho}(\tau, E) \tag{17}$$

where $\chi_{]-\infty,s]}$ is the **characteristic function** of $]-\infty, s]$. So that we obtain in the (ν, E)-representation the following expression of these spectral projection operators

$$\begin{aligned} \mathcal{P}_s \hat{\rho}(\nu, E) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{s} \mathrm{e}^{-\mathrm{i}\nu\tau} \hat{\rho}(\tau, E)\, \mathrm{d}\tau \\ &= \mathrm{e}^{-\mathrm{i}\nu s} \int_{-\infty}^{0} \mathrm{e}^{-\mathrm{i}\nu\tau} \hat{\rho}(\tau + s, E)\, \mathrm{d}\tau. \end{aligned} \tag{18}$$

Let us denote the Fourier transform $\mathcal{F}f(\nu) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\nu\tau} f(\tau)\, \mathrm{d}\tau$ and remind the Paley-Wiener theorem which says that a function $f(\nu)$ belongs to the Hardy class H^+(i.e., the limit as $y \to 0^+$ of an analytic function $\Phi(\nu + \mathrm{i}y)$ such that $\int_{-\infty}^{\infty} \mid \Phi(\nu + \mathrm{i}y) \mid^2 \, \mathrm{d}y < \infty$) if and only if it is of the form $f(\nu) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} \mathrm{e}^{-\mathrm{i}\nu\tau} \hat{f}(\tau)\, \mathrm{d}\tau$ where $\hat{f} \in L^2(\mathbb{R}^+)$ [13]. Using the Hilbert transformation

$$Hf(x) = \frac{1}{\pi} \mathsf{P} \int_{-\infty}^{\infty} \frac{f(t)}{t - x}\, \mathrm{d}t \tag{19}$$

for $f \in L^2(\mathbb{R})$ we can write the decomposition

$$f(x) = \frac{1}{2}[f(x) - \mathrm{i}Hf(x)] + \frac{1}{2}[f(x) + \mathrm{i}Hf(x)] = f_+(x) + f_-(x). \tag{20}$$

According to the theorem, $f_+(x)$ (respectively $f_-(x)$) belongs to the Hardy class H^+ (respectively H^-). This decomposition is unique as a result of Paley-Wiener theorem. Thus taking the Fourier transformation of f we obtain

$$\mathcal{F}(f)(\nu) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} \mathrm{e}^{-\mathrm{i}\nu\tau} \hat{f}(\tau)\, \mathrm{d}\tau + \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} \mathrm{e}^{-\mathrm{i}\nu\tau} \hat{f}(\tau)\, \mathrm{d}\tau.$$

It follows that

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0} \mathrm{e}^{-\mathrm{i}\nu\tau}\hat{f}(\tau)\,\mathrm{d}\tau = \frac{1}{2}(\mathcal{F}(f) - \mathrm{i}H\mathcal{F}(f)). \tag{21}$$

Now, using the well known property of the translated Fourier transformation $\sigma_s\hat{f}(\tau) = \hat{f}(\tau + s)$ we have

$$\mathcal{F}(\sigma_s\hat{f})(\nu) = \mathrm{e}^{\mathrm{i}\nu s}\mathcal{F}.\hat{f}(\nu) = \mathrm{e}^{\mathrm{i}\nu s}f(\nu) \tag{22}$$

this and (19) yields

$$\mathcal{P}_s\rho(\nu, E) = \frac{1}{2}\mathrm{e}^{-\mathrm{i}\nu s}[\mathrm{e}^{i\nu s}\rho(\nu, E) - \mathrm{i}H(\mathrm{e}^{\mathrm{i}\nu s}\rho(\nu, E))]. \tag{23}$$

Thus

$$\mathcal{P}_s\rho(\nu, E) = \frac{1}{2}[\rho(\nu, E) - \mathrm{i}\mathrm{e}^{-\mathrm{i}\nu s}H(\mathrm{e}^{\mathrm{i}\nu s}\rho(\nu, E))]. \tag{24}$$

It is clear from (1.15) that $\mathcal{P}_s\rho(\nu, E)$ is in the Hardy class H^+.

3. Computation of Spectral Projections of *T* in a Friedrichs Model

The one-level Friedrichs model is a simple model of hamiltonian in which a discrete eigenvalue the free Hamiltonian H_0. It has been often used as a simple model of decay of unstable states illustrating the Weiskopf-Wigner theory of decaying quantum systems. The Hamilton operator H is an operator on the Hilbert space of the wave functions of the form $|\psi>= \{f_0\,;\,g(\omega)\}, f_0 \in \mathbb{C}, g \in L^2(\mathbb{R}^+)$

$$H = H_0 + \lambda V \tag{25}$$

where λ is a positive coupling constant, and

$$H_0\,|\psi>= \{\omega_1 f_0\,;\,\omega g(\omega)\}, \quad \omega_1 > 0. \tag{26}$$

We shall denote the eigenfunction of H_0 by $\chi = \{1, 0\}$. The operator *V* is given by

$$V\{f, g(\omega)\} = \{<v(\omega), g(\omega)>, f_0.v(\omega)\}. \tag{27}$$

Thus H can be represented by the matrix

$$H = \begin{pmatrix} \omega_1 & \lambda v^*(\omega) \\ \lambda v(\omega) & \omega \end{pmatrix} \tag{28}$$

where $v(\omega) \in L^2(\mathbb{R}^+)$ and reffered as a factor form. It has been shown than for enough small λ, H has no eigenvalues and that the spectrum of H can be continuously extended over $\mathbb{R}^+$. It is also shown that in the outgoing spectral representation of H, the vector χ is represented by

$$f_1(\omega) = \frac{\lambda v(\omega)}{\eta^+(\omega + \mathrm{i}\epsilon)} \tag{29}$$

where

$$\eta^{+}(\omega+\mathrm{i}\epsilon)=\omega-\omega_1+\lambda^2\lim_{\epsilon\to 0}\int_0^{\infty}\frac{|v(\omega)|^2}{\omega'-\omega-\mathrm{i}\epsilon}\mathrm{d}\omega' \tag{30}$$

and $H\chi$ is represented by $\omega f_1(\omega)$. The quantity $<\chi,\mathrm{e}^{-\mathrm{i}Ht}\chi>$ is usually called the decay law and $|<\chi,\mathrm{e}^{-\mathrm{i}Ht}\chi>|^2=\int_0^{\infty}|f_1(\omega)|^2\mathrm{e}^{-\mathrm{i}\omega t}d\omega$ is called the survival probability at time t. It is however clear that this is not a true probability, since it is not a monotonically decreasing quantity, although it tends to zero as a result of the Riemann-Lebesgue lemma. Let us now identify the state χ with element $\rho=|\chi><\chi|$ of the Liouville space, that is, to the kernel operator

$$\rho_{11}(\omega,\omega')=f_1(\omega)\overline{f_1(\omega')}. \tag{31}$$

We shall compute first the unstable component $\mathcal{P}_0\rho_{11}$ and show that $\mathcal{P}_0\rho_{11}\neq\rho_{11}$. Then we shall compute the survival probability in the state ρ

$$\lim_{s\to\infty}||\mathcal{P}_{-s}\rho||^2\to 0. \tag{32}$$

4. Computation of $\mathcal{P}_s\rho_{11}$

As explained above the Liouville operator is given by

$$L\rho(\omega,\omega')=(\omega-\omega')\rho(\omega,\omega') \tag{33}$$

and that the spectral representation of L is given by the change of variables

$$\nu=\omega-\omega' \tag{34}$$

and

$$E=\min(\omega,\omega'). \tag{35}$$

Thus we obtain for $\rho_{11}(\nu,E)$

$$\rho_{11}(\nu,E)=\begin{cases}\lambda^2\frac{v(E)}{\eta^-(E)}\frac{v^*(E+\nu)}{\eta^+(E+\nu)}, & \nu>0\\ \\ \lambda^2\frac{v^*(E)}{\eta^+(E)}\frac{v(E-\nu)}{\eta^-(E-\nu)}, & \nu<0\end{cases} \tag{36}$$

where η^- is the complex conjugate of η^+ and

$$\eta^+(\omega)\simeq\omega-z_1,\qquad z_1=\tilde{\omega}_1-\mathrm{i}\frac{\gamma}{2} \tag{37}$$

where z_1 is called the resonance with energy $\tilde{\omega}_1$ and a lifetime γ [10]. It is believed that this form results from weak coupling approximations. It can be shown $\rho_{11}(\nu,E)$ in the following form

$$\rho_{11}(\nu,E)=\frac{\gamma}{2}f(\nu) \tag{38}$$

where

$$f(\nu) = \begin{cases} \frac{1}{\nu_0^*(\nu+\nu_0)}, & \nu > 0 \\ \frac{1}{\nu_0(\nu_0^*-\nu)}, & \nu < 0 \end{cases} \tag{39}$$

and where $\nu_0 = a + \mathrm{i}b = (E - \widetilde{\omega}_1) + \mathrm{i}\frac{\gamma}{2}$. For obtaining $\mathcal{P}_s(f)(\nu)$, we shall use the formula (24) and thus

$$\begin{aligned} \mathcal{P}_s f(\nu) = & \ \mathrm{i}\mathrm{e}^{-\mathrm{i}s\nu}[\frac{-1}{2\pi\nu_0(\nu_0^* - \nu)}(\int_{-\infty}^{0} \frac{\mathrm{e}^{-sy}}{y + \mathrm{i}\nu_0^*}\mathrm{d}y - \int_{-\infty}^{0} \frac{\mathrm{e}^{-sy}}{y + \mathrm{i}\nu}\mathrm{d}y) \\ & + \frac{1}{2\pi\nu_0^*(\nu + \nu_0)}(\int_{-\infty}^{0} \frac{\mathrm{e}^{-sy}}{y - \mathrm{i}\nu_0}\mathrm{d}y - \int_{-\infty}^{0} \frac{\mathrm{e}^{-sy}}{y + \mathrm{i}\nu}\mathrm{d}y)] \\ & + \begin{cases} \mathrm{e}^{-\mathrm{i}s\nu}[\frac{\mathrm{e}^{\mathrm{i}s\nu_0^*}}{\nu_0(\nu_0^*-\nu)} - \frac{\mathrm{e}^{-\mathrm{i}s\nu_0}}{\nu_0^*(\nu_0+\nu)}], & E < \widetilde{\omega}_1 \\ 0, & E > \widetilde{\omega}_1. \end{cases} \end{aligned} \tag{40}$$

We can also compute the same result for the case $\nu < 0$.

4.1. Case $s = 0$

In this case (40) cab be obtained as

$$\begin{aligned} \mathcal{P}_0 f(\nu) = & \ \frac{\mathrm{i}}{\nu_0(\nu_0^* - \nu)} \log^+(\frac{\nu}{\nu_0^*}) - \frac{\mathrm{i}}{\nu_0^*(\nu + \nu_0)} \log^+(-\frac{\nu}{\nu_0}) \\ & + \begin{cases} [\frac{1}{\nu_0(\nu_0^*-\nu)} - \frac{1}{\nu_0^*(\nu_0+\nu)}], & E < \widetilde{\omega}_1 \\ 0, & E > \widetilde{\omega}_1 \end{cases} \end{aligned} \tag{41}$$

where $\log^+ z$ is the complexe analytic function with cut-line along the negative axis

$$\log^+ z = \log|z| + \mathrm{i}\arg(z), \qquad \arg(z) \in] - \frac{\pi}{2}, \frac{3\pi}{2}[\,. \tag{42}$$

Also, we have used $\lim_{R\to\infty} \log^+(\frac{\mathrm{i}\nu - R}{\mathrm{i}\nu_0^* - R}) \to 0$ and $\lim_{R\to\infty} \log^+(\frac{\mathrm{i}\nu - R}{-\mathrm{i}\nu_0 - R}) \to 0$.

We see that $\mathcal{P}_0 f(\nu)$ is an upper Hardy class function. This verified the general theorem about the properties of the unstable states associated to time operator, as being in the upper Hardy class.

4.2. Asymptotic of the Survival Probability

First, using the following approximation, for $s \to -\infty$

$$\int_{-\infty}^{0} \frac{e^{-sz}}{y+z} dy = e^{sx} \int_{-\infty}^{z} \frac{e^{-su}}{u} du = e^{sz} \Big\{ \Big[\frac{e^{-su}}{-su} \Big]_{-\infty}^{z} - \int_{-\infty}^{z} \frac{e^{-su}}{su^2} du \Big\}$$
$$= \frac{1}{(-zs)} \Big[1 + \frac{1}{(-zs)} + \frac{2!}{(-zs)^2} + \cdots + \frac{n!}{(-zs)^n} + r_n(-zs) \Big] \tag{43}$$

where the last result was obtained by integral part by part repetitions, z can be a complex number, and

$$r_n(z) = (n+1)! z e^{-z} \int_{-\infty}^{z} \frac{e^t}{t^{n+2}} dt \tag{44}$$

so we have [9]

$$|r_n(z)| \leq \frac{(n+1)!}{|z|^{n+1}}. \tag{45}$$

Thus, by using the above approximation in the equations (40) and (38) in the limit $s \to -\infty$ we obtain an estimate of the decay probability

$$\int_0^{\infty} \int_{-\infty}^{+\infty} |\mathcal{P}_s \rho_{11}(\nu, E)|^2 d\nu dE \leq \frac{\gamma^2}{4} [\frac{h(\gamma, \tilde{\omega}_1)}{\gamma^4 s^4} + e^{\gamma s} h_1(s, \gamma, \tilde{\omega}_1)] \tag{46}$$

where

$$h(\gamma, \tilde{\omega}_1) = (\frac{256}{\pi\gamma^2})[\frac{7\pi}{64} + \frac{7}{32} \arctan \frac{2\tilde{\omega}_1}{\gamma} - \frac{1}{12} \sin^3(2 \arctan \frac{2\tilde{\omega}_1}{\gamma})$$
$$+ \frac{1}{4} \sin(2 \arctan \frac{2\tilde{\omega}_1}{\gamma}) - \frac{1}{16} \sin(4 \arctan \frac{2\tilde{\omega}_1}{\gamma}) + \frac{1}{256} \sin(8 \arctan \frac{2\tilde{\omega}_1}{\gamma})] \tag{47}$$

and

$$h_1(s, \gamma, \tilde{\omega}_1) = 2[\frac{\pi}{\gamma} \arctan \frac{2\tilde{\omega}_1}{\gamma} + \frac{\gamma \sin(2\tilde{\omega}_1 s) - 2\tilde{\omega}_1 \cos(2\tilde{\omega}_1 s)}{s(\tilde{\omega}_1^2 + \frac{\gamma^2}{4})}] \tag{48}$$

Here we have an algebraically decreasing function and an exponentially decreasing multiply by the oscillating functions.

5. Conclusion

We have shown that the pure initial state $\rho(t) = |\psi_t\rangle\langle\psi_t|$, decomposes into decaying state and a background, $\rho(t) \to \mathcal{P}_0\rho(t) + (1 - \mathcal{P}_0\rho(t))$. In the other hand, our result shows that the survival probability is decreasing for long time exponentially and algebraically, i.e., we do not have a Zeno effect for our survival probability.

Recently, we have studied two-level Friedrichs model with weak coupling interaction constants for a decay phenomena in the Hilbert space for kaonic system [4,5]. In future, we shall consider two-level or n-level Friedrichs by using Time Super-Operator in the Liouville space to study in order an irreversible decay description.

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