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THE VARIATIONAL PRINCIPLE OF HERGLOZ AND RELATED RESULTS

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Abstract. This is a review of the variational principle proposed by Gustav Hergotz and resent results related to it. In that variational principle the functional is defined by a certain differential equation instead of an integral. The solutions of the equations for the extrema of the functional determine contact transformations. Some of those results are: two Noether-type theorems for finding conserved quantities and identities, a method for calculating symmetry groups of the functional and several applications.

1. Introduction

In the 1930-s Gustav Herglotz proposed a *generalized variational principle* with one independent variable, which generalizes the classical variational principle by defining the functional, whose extrema are sought, by a certain ordinary differential equation. Herglotz variational principle contains the classical variational principle as a special case. His original idea was published in 1979 in his collected works [8] and [9]. It is especially suitable for a variational description of nonconservative processes. It can give a variational description of such processes even when the Lagrangian is not dependent on time, something which can not be done with the classical variational principle. It is also closely related to contact transformations.

The generalized variational principle of Herglotz defines the functional z, whose extrema are sought, by the differential equation

$$\frac{\mathrm{d}z}{\mathrm{d}t} = L\Big(t, x(t), \frac{\mathrm{d}x(t)}{\mathrm{d}t}, z\Big) \tag{1}$$

where t is the independent variable, and $x(t) \equiv (x_1(t), \ldots, x_n(t))$ stands for the argument functions. In order for the equation (1) to define a functional $z = z[x]$ of $x(t)$ it has to be solved with the same fixed initial condition $z(0)$ for all argument functions $x(t)$, and the solution $z(t)$ must be evaluated at the same fixed final time $t = T$ for all argument functions $x(t)$.

The equations whose solutions produce the extrema of this functional are

$$\frac{\partial L}{\partial x_k} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}_k} + \frac{\partial L}{\partial z}\frac{\partial L}{\partial \dot{x}_k} = 0\,, \qquad k = 1, \ldots, n \tag{2}$$

where $\dot{x}_k$ denotes $\mathrm{d}x_k/\mathrm{d}t$. Herglotz called them the **generalized Euler-Lagrange equations**. See Guenther *et al* [7] for a derivation of this system.

Dissipation and generation effects in physical processes can often be accounted for in the equations describing these processes by terms which are proportional to the first time derivatives $\dot{x}_i(t) = \mathrm{d}x_i/\mathrm{d}t$ of the dependent variables (see Goldstein [6]). For example, the viscous frictional forces acting on an object which is moving in a resistive medium, such as a gas or a liquid, are proportional to the object's velocity. Similarly, the dissipative effects (due to the ohmic resistance) in electrical circuits can often be modeled by including terms which are proportional to the first time-derivative of the corresponding dependent variables, such as the electric charge.

All such dissipative processes can be given a unified description by the generalized variational principle.

For example, let us consider the motion of a small object with mass m (point mass) under the action of some potential $U = U(x)$ with $x = (x_1, x_2, x_3)$ in a resistive medium. We assume that the resistive forces are proportional to the velocity. Then the equations describing the motion of such an object are

$$m\,\ddot{x}_i = -\frac{\partial U}{\partial x_i} - k\,\dot{x}_i\,, \qquad i = 1, 2, 3 \tag{3}$$

where $k > 0$ is a constant and and $\cdot$ denotes differentiation with respect to t. All equations of this form can be obtained from the generalized variational principle by choosing for the Lagrangian function L the expression

$$L = \frac{m}{2}(\dot{x}_1^2 + \cdots + \dot{x}_n^2) - U(x_1, \ldots, x_n) - \alpha\, z \tag{4}$$

where $U = U(x_1, \ldots, x_n)$ is the potential energy of the system and $\alpha > 0$ is a constant. From (4) we obtain the generalized Euler-Lagrange equations

$$\frac{\partial L}{\partial x_i} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}_i}\right) + \frac{\partial L}{\partial z}\,\frac{\partial L}{\partial \dot{x}_i} = -\frac{\partial U}{\partial x_i} - \frac{\mathrm{d}}{\mathrm{d}t}(m\dot{x}_i) - m\alpha\,\dot{x}_i = 0$$

which are the same as (3) for $n = 3$ and $k = m\alpha$.

Depending on the choice of the function U, equations (3) can describe a variety of systems. For instance

1. When $U = cr^2 = c(x_1^2 + \cdots + x_n^2)$, with $c > 0$ constant, equations (3) describe multi-dimensional **isotropic damped harmonic oscillators**.

2. When $U = -c/r = -c/\sqrt{x_1^2 + x_2^2 + x_3^2}$, equations (3) describe the motion of a point mass m under Coulomb (electrostatic) or gravitational forces in a resistive medium characterized by the constant α.
3. The equations describing the currents, voltages and charges in single or coupled electrical circuits have the same form as equations (3) with $i = 1, \ldots, n$, where n is the number of **state variables** (currents, voltages and charges) and U is an appropriately chosen function (see Goldstein [6], p. 52). Hence, the processes in electrical circuits can also be derived from a Lagrangian function of the form (4) via the generalized variational principle. The interested reader can find more on this topic in Georgieva et al. [2].

Further we use the generalized variational principle of Herglotz to give a variational description of several named ordinary differential equations.

First we show that the class of ordinary differential equations

$$\ddot{x} + f(x)\dot{x}^2 + g(t)\dot{x} + h(x) = 0 \tag{5}$$

for the function $x = x(t)$ can be given a variational description via the Herglotz variational principle, by letting L in the defining equation (1) be

$$L = \frac{1}{2}\dot{x}^2 - (2f(x)\dot{x} + g(t))z - U(x)$$

where $U(x)$ is any solution of the ODE

$$\frac{\mathrm{d}U(x)}{\mathrm{d}x} + 2f(x)U(x) = h(x).$$

Equation (5) contains several well known named equations as special cases

a. When $h(x) = kx$, with $k = \text{const}$, $f(x) = 0$ and $g(t) = a = \text{const}$, equation (5) is the equation of the damped harmonic oscillator

$$\ddot{x} + a\dot{x} + kx = 0.$$

The corresponding Lagrangian is

$$L = \frac{1}{2}(\dot{x}^2 - kx^2) - az.$$

b. In the case when $h(x) = x^n$, $f(x) = 0$ and $g(t) = 2/t$, equation (5) becomes the Lane-Emden equation

$$\ddot{x} + \frac{2}{t}\dot{x} + x^n = 0, \qquad n \neq -1.$$

In that case the Lagrangian is

$$L = \frac{1}{2}\dot{x}^2 - \frac{x^{n+1}}{n+1} - \frac{2}{t}z.$$

c. As a final example consider the special case when $h(x) = 0$. Then equation (5) is the Liouville's equation

$$\ddot{x} + f(x)\dot{x}^2 + g(t)\dot{x} = 0.$$

The Lagrangian for it is

$$L = \frac{1}{2}\dot{x}^2 - \big(2f(x)\dot{x} + g(t)\big)z.$$

These and other examples of using the Herglotz functional to give a variational description of non-conservative ordinary differential equations can be found in Georgieva [5].

2. Contact Transformations and Their Relation to Herglotz Principle

The solutions of the generalized Euler-Lagrange equations, when written in terms of the dependent variables x_k and the associated momenta $p_k = \partial L/\partial \dot{x}_k$, determine a family of *contact transformations*. In the present section we will prove this remarkable fact, which has both theoretical and applied significance. The proofs of the theorems in this section as well as a detailed treatment of contact transformations can be found in Guenther *et al* [7].

First we will recall a few facts about contact transformations.

Consider surfaces $z = f(x,y)$ in $\mathbb{R}^3$. Denote $p = \frac{\partial f}{\partial x}$, $q = \frac{\partial f}{\partial y}$.

Definition 1. (x, y, z, p, q) *is called a surface element if* $(-p, -q, 1)$ *is orthogonal to the tangent plane to the surface at the point* (x, y, z)*, i.e.,* $p\,\mathrm{d}x + q\,\mathrm{d}y - \mathrm{d}z = 0$.

Consider the transformation T of the (x, y, z, p, q)-space to the (X, Y, Z, P, Q)-space defined by $X = X(x, y, z, p, q)$, $Y = Y(x, y, z, p, q)$, $Z = Z(x, y, z, p, q)$, $P = P(x, y, z, p, q)$, $Q = Q(x, y, z, p, q)$.

Definition 2. *Let T be a one-to-one, onto, continuously differentiable transformation of the (x, y, z, p, q)-space to the (X, Y, Z, P, Q)-space with a nonzero Jacobian. Then T is called a* ***contact transformation*** *if* $p\,\mathrm{d}x + q\,\mathrm{d}y - \mathrm{d}z = 0$ *implies* $P\,\mathrm{d}X + Q\,\mathrm{d}Y - \mathrm{d}Z = 0$.

Theorem 1. *The one-to-one, onto, continuously differentiable transformation T of the (x, y, z, p, q)-space to the (X, Y, Z, P, Q)-space with a nonzero Jacobian is a contact transformation if and only if there exists a nonzero function $\rho = \rho(x, y, z, p, q)$ such that*

$$P\,\mathrm{d}X + Q\,\mathrm{d}Y - \mathrm{d}Z = \rho(p\,\mathrm{d}x + q\,\mathrm{d}y - \mathrm{d}z). \qquad (6)$$

Example 1. The Legendre transformation $X = p$, $Y = q$, $P = x$, $Q = y$, $Z = px + qy - z$ is a contact transformation. A necessary and sufficient condition for the function ρ in the previous theorem is $\rho = -1$.

Let now $x = (x_1, \dots, x_n)$, $p = (p_1, \dots, p_n)$ and S_t be a one-parameter family of contact transformations

$$X = X(x, z, p, t), \qquad Z = Z(x, z, p, t), \qquad P = P(x, z, p, t)$$

where t is the parameter, $X = (X_1, \dots, X_n)$ stands for the images of $x_1, \dots, x_n$ under S_t, Z is the image of z under S_t and $P = (P_1, \dots, P_n)$ stands for the images of $p_1, \dots, p_n$ under S_t.

The summation convention on repeated indicesis is used in the entire paper.

For one-parameter families of contact transformations the necessary and sufficient condition (6) for a contact transformation is replaced by

$$P_i \, \mathrm{d}X_i - \mathrm{d}Z = \rho\big(p_i \, \mathrm{d}x_i - \mathrm{d}z\big) + H\mathrm{d}t \tag{7}$$

where

$$H = P_i \frac{\partial X_i}{\partial t} - \frac{\partial Z}{\partial t} \cdot$$

We will show below that the solutions of the generalized Euler-Lagrange equations (2), when written in terms of the dependent variables x_k and the associated momenta $p_k = \partial L / \partial \dot{x}_k$, determine a family of *contact transformations*. For this let us write the defining equation (1) for the functional z and the generalized Euler-Lagrange equations (2) in the following manner

$$\begin{aligned} \dot{z} &= L(x_1, \dots, x_n, \dot{x}_1, \dots, \dot{x}_n, z, t) \\ \dot{p}_j &= L_j + L_z p_j, \qquad j = 1 \dots, n \end{aligned} \tag{8}$$

where we have denoted

$$\frac{\partial L}{\partial x_j} = L_j, \qquad \frac{\partial L}{\partial \dot{x}_j} = p_j.$$

Let $(x^0, \dot{x}^0, \dot{z}^0)$ be the initial condition for the system (8) of $n + 1$ ordinary differential equations for the functions $x_1(t), \dots, x_n(t)$, $z(t)$. Then the solution of the system (8) with this initial condition is

$$x = x(x^0, \dot{x}^0, \dot{z}^0, t), \qquad \dot{x} = \dot{x}(x^0, \dot{x}^0, \dot{z}^0, t), \qquad z = z(x^0, \dot{x}^0, \dot{z}^0, t). \tag{9}$$

Theorem 2. *Let* $L = L(x, \dot{x}, z, t)$ *be such that*

$$\det\left(\frac{\partial^2 L}{\partial \dot{x}_i \partial \dot{x}_j}\right) \neq 0.$$

Then the solution of the system (8) *defines a one-parameter family of contact transformations.*

Proof: Let us differentiate the first equation in (8) with respect to the t-variable to obtain

$$\ddot{z} = L_j \dot{x}_j + L_z \dot{z} + p_j \ddot{x}_j + L_t \,.$$

Let us denote also

$$\lambda = \exp\Big(-\int_0^t \frac{\partial L}{\partial z} \mathrm{d}\theta\Big). \tag{10}$$

Then

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(\lambda \dot{z}) &= \lambda\big(L_j \dot{x}_j + p_j \ddot{x}_j\big) + \lambda L_t + \lambda L_z \dot{z} + \dot{\lambda}\dot{z} \\
&= \lambda L_j \dot{x}_j - \dot{x}_j \frac{\mathrm{d}}{\mathrm{d}t}(\lambda p_j) + \frac{\mathrm{d}}{\mathrm{d}t}(\lambda p_j \dot{x}_j) + \lambda L_t \\
&= \lambda L_j \dot{x}_j - \dot{x}_j\big(-\lambda L_z p_j + \lambda(L_j + L_z p_j)\big) + \frac{\mathrm{d}}{\mathrm{d}t}(\lambda p_j \dot{x}_j) + \lambda L_t \\
&= \frac{\mathrm{d}}{\mathrm{d}t}(\lambda p_j \dot{x}_j) + \lambda L_t.
\end{aligned}$$

Therefore

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\lambda\big(\dot{z} - p_j \dot{x}_j\big)\Big) = \lambda L_t.$$

Next we integrate the last equation with respect to t, to obtain

$$\lambda\big(\dot{z} - p_j \dot{x}_j\big) - \big(\dot{z}^0 - p_j^0 \dot{x}_j^0\big) = \int_0^t \lambda L_\theta \,\mathrm{d}\theta$$

or equivalently

$$p_j \dot{x}_j - \dot{z} = \frac{1}{\lambda}\Big(p_j^0 \dot{x}_j^0 - \dot{z}^0\Big) - \frac{1}{\lambda}\int_0^t \lambda L_\theta \,\mathrm{d}\theta. \tag{11}$$

Let us denote with

$$H = -\frac{1}{\lambda}\int_0^t \lambda L_\theta \,\mathrm{d}\theta \qquad \text{and} \qquad \rho = \frac{1}{\lambda}\cdot$$

Then equation (11) can be written as

$$p_j \mathrm{d}x_j - \mathrm{d}z = \rho\Big(p_j^0 \mathrm{d}x_j^0 - \mathrm{d}z^0\Big) + H\,\mathrm{d}t$$

which is the necessary and sufficient condition for the transformation (9), depending on the parameter t, to be a contact transformation. In order to write the transformation (9) with the p_j-s instead of the $\dot{x}_j$-s , we recall the relations

$$p_j = \frac{\partial L}{\partial \dot{x}_j}(x, \dot{x}, z, t), \qquad j = 1, \ldots, n$$

and solve this system of n equations for the $\dot{x}_j$-s as functions of the p_j-s. This last step is justified by the hypothesis that the Hessian of L is nonzero. □

3. First Noether-type Theorem

In this section we discuss a theorem which gives a method for a systematic derivation of conserved quantities for equations which have a variational description via Herglotz variational principle.

Consider the one-parameter group of transformations

$$\bar{t} = \phi(t, x, \varepsilon), \qquad \bar{x}_k = \psi_k(t, x, \varepsilon), \qquad k = 1, ..., n \tag{12}$$

where ε is the parameter, $\phi(t, x, 0) = t$, and $\psi_k(t, x, 0) = x_k$, with infinitesimal generator

$$\mathbf{v} = \tau(t, x)\frac{\partial}{\partial t} + \xi_k(t, x)\frac{\partial}{\partial x_k}$$

where

$$\tau(t, x) = \frac{\mathrm{d}\phi}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0} \qquad \text{and} \qquad \xi_k(t, x) = \frac{\mathrm{d}\psi_k}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0}. \tag{13}$$

Theorem 3 (First Noether-type theorem for the generalized variational principle). *If the functional* $z = z[x(t)]$ *defined by the differential equation* $\dot{z} = L(t, x, \dot{x}, z)$ *is invariant under the one-parameter group of transformations* (12) *then the quantity*

$$\exp\Big(-\int_0^t \frac{\partial L}{\partial z}\,\mathrm{d}\theta\Big)\Big(\Big(L - \dot{x}_k\frac{\partial L}{\partial \dot{x}_k}\Big)\tau + \frac{\partial L}{\partial \dot{x}_k}\xi_k\Big) \tag{14}$$

is conserved along the solutions of the generalized Euler-Lagrange equations (2).

The conserved quantities (14) have a remarkable form - they are products of λ defined in (10) with the expressions

$$\Big(L - \dot{x}_k\frac{\partial L}{\partial \dot{x}_k}\Big)\tau + \frac{\partial L}{\partial \dot{x}_k}\xi_k$$

whose form is exactly the same as that of the conserved quantities obtained from the classical first Noether theorem. In the special case $\partial L/\partial z = 0$ the functional z is defined by the integral

$$z = \int_0^t L(t, x, \dot{x})\,\mathrm{d}\theta$$

and $\lambda = 1$. Hence, in that case Theorem 3 reduces to the classical first Noether theorem.

We are now ready to apply the first Noether-type theorem to find specific conserved quantities corresponding to several basic symmetries. Because of their generality and physical significance we state the results as corollaries to the first Noether-type theorem.

Corollary 1. *If we start with the functional z defined by the differential equation $\dot{z} = L(t, x, \dot{x}, z)$ which is invariant with respect to the translations in time, $\bar{t} = t + \varepsilon$, $\bar{x} = x$ the quantity*

$$E = \exp\Big(-\int_0^t \frac{\partial L}{\partial z}\,\mathrm{d}\theta\Big)\Big(L(x, \dot{x}, z) - \frac{\partial L}{\partial \dot{x}_k}\dot{x}_k\Big)$$

is conserved on solutions of the generalized Euler-Lagrange equations.

Corollary 2. *If we start with the functional z defined by the differential equation $\dot{z} = L(t, x, \dot{x}, z)$ which is invariant with respect to the translations in space direction x_k, i.e, $\bar{t} = t$, $\bar{x}_k = x_k + \varepsilon$, $\bar{x}_i = x_i$ for $i = 1, ..., k-1, k+1, ..., n$ the quantity*

$$M_k = \exp\Big(-\int_0^t \frac{\partial L}{\partial z}\,\mathrm{d}\theta\Big)\frac{\partial L}{\partial \dot{x}_k}$$

is conserved on solutions of the generalized Euler-Lagrange equations.

Corollary 3. *Let the functional z defined by the equation $\dot{z} = L(t, x, \dot{x}, z)$ is invariant with respect to the rotations in the $x_i x_j$-plane. Then the quantity*

$$A_{ij} = \exp\Big(-\int_0^t \frac{\partial L}{\partial z}\,\mathrm{d}\theta\Big)\Big(\frac{\partial L}{\partial \dot{x}_i}x_j - \frac{\partial L}{\partial \dot{x}_j}x_i\Big)$$

is conserved along solutions of the generalized Euler-Lagrange equations.

The proof of Theorem 3 can be found in Georgieva *et al* [2].

4. Variational Symmetries of the Herglotz Functional

Physical systems described by the generalized Euler-Lagrange equations (2) are not conservative in general. Below we show how the first Noether-type theorem can be used to find conserved quantities in non-conservative systems. For this, we must describe the physical system with the generalized Euler-Lagrange equations and then find symmetries of the functional $z = z[x(t); t]$ defined by the differential equation $\dot{z} = L(t, x, \dot{x}, z)$, that is, transformations of both dependent and independent variables which leave $z[x(t); t]$ invariant.

In order to apply the first Nother-type theorem to find conserved quantities for the system in consideration, one needs to know a one-parameter group of symmetries of the Herglotz functional z. In this section we discuss a method for calculating such symmetries.

Historically, the question of calculating the symmetries of a given Lagrangian functional (defined by an integral) was answered by W. Killing [10] in 1892 in the context of describing the motions of a n-dimensional manifold with fundamental form given by

$$L = \frac{1}{2} g_{kl} \dot{x}^k \dot{x}^l$$

(see Eisenhart [1] and Logan [12]). In the case of a classical variational functional, some authors refer to the system of partial differential equations for the unknown symmetry group generators as the *generalized Killing equations*.

The following theorem gives a method for finding symmetry groups of the Herglotz functional.

Consider one-parameter families of transformations of the independent variable t and the dependent variables $x_1, \ldots, x_n$, like in (12).

Theorem 4. *The coefficients $\tau(t,x)$ and $\xi_k(t,x)$ of the infinitesimal generator of a one-parameter group which preserve the value of the functional $z = z[x(t)]$, defined by the differential equation* (1)*, are solutions to the system of partial differential equations obtained from the identity*

$$\frac{\partial L}{\partial t}\tau + \frac{\partial L}{\partial x_k}\xi_k + \frac{\partial L}{\partial \dot{x}_k}\Big(\frac{\partial \xi_k}{\partial t} + \frac{\partial \xi_k}{\partial x_j}\dot{x}_j - \dot{x}_k\frac{\partial \tau}{\partial t} - \dot{x}_k\dot{x}_j\frac{\partial \tau}{\partial x_j}\Big) + L\Big(\frac{\partial \tau}{\partial t} + \frac{\partial \tau}{\partial x_j}\dot{x}_j\Big) = 0$$

by equating to zero the coefficients in front of the powers of z and $\dot{x}_k$ and in front of products of such powers.

In analogy with the classical case, we call this identity the *fundamental invariance identity* and the resulting partial differential equations for the coefficients of the infinitesimal generator of the variational symmetry group the *generalized Killing equations*.

Next we show an example of calculating a variational symmetry group with this method, by applying it to the Liouville's equation $\ddot{x} + f(x)\dot{x}^2 + g(t)\dot{x} = 0$ with a specific choice of the coefficient functions, namely

$$f(x) = \frac{h}{kx + a}, \qquad g(t) = \frac{c}{2kt + b} \tag{15}$$

where a, b, c, h, and k are arbitrary constants (if $k = 0$ then a and b must be nonzero). As noted earlier, this equation can be given a variational description via the Herglotz variational principle if the functional z is defined by the differential equation

$$\dot{z} = \frac{1}{2}\dot{x}^2 - (2f(x)\dot{x} + g(t))z.$$

The fundamental invariance identity of Theorem 4 takes the form

$$-\frac{\mathrm{d}g}{\mathrm{d}t}z\tau - 2\frac{\mathrm{d}f}{\mathrm{d}x}\dot{x}z\xi + (\dot{x} - 2f(x)z)\Big(\frac{\partial\xi}{\partial t} + \frac{\partial\xi}{\partial x}\dot{x} - \dot{x}\frac{\partial\tau}{\partial t} - \dot{x}^2\frac{\partial\tau}{\partial x}\Big) + \Big(\frac{1}{2}\dot{x}^2 - 2f(x)\dot{x}z - g(t)z\Big)\Big(\frac{\partial\tau}{\partial t} + \frac{\partial\tau}{\partial x}\dot{x}\Big) = 0.$$

With the specific choices (15) for $f(x)$ and $g(t)$ the system of partial differential equations obtained from this identity after equating to zero the proper coefficients has the solutions $\xi = kx + a$ and $\tau = 2kt + b$. Thus, the variational symmetry of the Liouville's equation produced by this method is

$$\bar{x} = x + (kx + a)\varepsilon, \qquad \bar{t} = t + (2kt + b)\varepsilon.$$

The corresponding conserved quantity of the Liouville's equation, obtained through an application of the first Noether-type Theorem 3, is

$$Q = \Big(\frac{kx(t) + a}{kx(0) + a}\Big)^{2h/k}\Big(\frac{2kt + b}{b}\Big)^{c/2k}\Big(\dot{x}(kx + a) - (2kt + b)\frac{\dot{x}^2}{2} - (2h + c)z\Big).$$

This method also produces a variational symmetry for the equation

$$\ddot{x} + \frac{2}{t}\dot{x} + \frac{1}{x^3} = 0.$$

In this case the functional z is defined by the equation

$$\dot{z} = \frac{1}{2}\dot{x}^2 + \frac{1}{2x^2} - \frac{2}{t}z$$

and the fundamental invariance identity of Theorem 4 assumes the form

$$2\frac{1}{t^2}z\tau - \frac{1}{x^3}\xi + \dot{x}\Big(\frac{\partial\xi}{\partial t} + \frac{\partial\xi}{\partial x}\dot{x} - \dot{x}\frac{\partial\tau}{\partial t} - \dot{x}^2\frac{\partial\tau}{\partial x}\Big) + \Big(\frac{1}{2}\dot{x}^2 + \frac{1}{2x^2} - \frac{2}{t}z\Big)\Big(\frac{\partial\tau}{\partial t} + \frac{\partial\tau}{\partial x}\dot{x}\Big) = 0.$$

The system of PDE's for the coefficients $\tau(t,x)$ and $\xi(t,x)$ of the infinitesimal generator of the variational symmetry group has the solution $\tau = 2kt$, $\xi = kx$, where k is an arbitrary constant. The corresponding conserved quantity is

$$Q = -kt^2\Big(\Big(\dot{x}^2 - \frac{1}{x^2}\Big)t - x\dot{x} + 4z\Big).$$

The proof of Theorem 4 as well as the above and other examples can be found in Georgieva [5].

5. Second Noether-type Theorem for the Generalized Variational Principle of Herglotz

The following theorem extends the classical second theorem of Emmy Noether so that it applies to the generalized variational principle of Herglotz. This theorem, which we called the *second Noether-type theorem*, provides an identity for each infinite–dimensional symmetry group of the functional $z[x; s]$ as defined by equation (1).

Theorem 5. *Let the infinite-dimensional group of transformations*

$$\begin{aligned} \bar{t} &= \phi(t, x, p(t), p^{(1)}(t), \ldots, p^{(r)}(t)) \\ \bar{x}_k &= \psi_k(t, x, p(t), p^{(1)}(t), \ldots, p^{(r)}(t)), \qquad k = 1, \ldots, n \end{aligned} \tag{16}$$

which depends on the function $p(t) \in \mathrm{C}^{r+2}$ *and its derivatives* $p^{(i)} = \mathrm{d}^i p/\mathrm{d}t^i$, *subject to the conditions* $\bar{t} = t$ *and* $\bar{x}_k = x_k$ *if* $p(t) = p^{(1)}(t) = \ldots = p^{(r)}(t) = 0$, *be a symmetry group of the functional* $z[x; s]$ *defined by the differential equation* (1). *Then the identity*

$$\tilde{X}_k(E\,Q^k) - \tilde{U}(E\,Q^k\,\dot{x}_k) = 0 \tag{17}$$

holds. Here $\tilde{U}$ *and* $\tilde{X}_k$ *are the adjoints of the linear differential operators*

$$\begin{aligned} U &= \frac{\partial \phi}{\partial p} + \frac{\partial \phi}{\partial p^{(1)}}\frac{\mathrm{d}}{\mathrm{d}t} + \ldots + \frac{\partial \phi}{\partial p^{(r)}}\frac{\mathrm{d}^r}{\mathrm{d}t^r} \\ X_k &= \frac{\partial \psi_k}{\partial p} + \frac{\partial \psi_k}{\partial p^{(1)}}\frac{\mathrm{d}}{\mathrm{d}t} + \ldots + \frac{\partial \psi_k}{\partial p^{(r)}}\frac{\mathrm{d}^r}{\mathrm{d}t^r}, \qquad k = 1, \ldots, n \end{aligned}$$

evaluated at $p(t) = p^{(1)}(t) = \ldots = p^{(r)}(t) = 0$, Q^k *denote the generalized Euler-Lagrange expressions*

$$Q^k = \frac{\partial L}{\partial x_k} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}_k} + \frac{\partial L}{\partial z}\frac{\partial L}{\partial \dot{x}_k}, \qquad k = 1, \ldots, n \qquad \textit{and}$$

$$E = \exp\Big(-\int_0^t \frac{\partial L}{\partial z}\,\mathrm{d}\theta\Big).$$

Observe that $E = 1$ if L does not depend on z. Then the identity (17) reduces to the identity provided by the classical second Noether theorem, namely

$$\tilde{X}_k\Big(\frac{\partial L}{\partial x_k} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}_k}\Big) - \tilde{U}\Big(\Big(\frac{\partial L}{\partial x_k} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}_k}\Big)\dot{x}_k\Big) = 0.$$

Thus, we see that when the generalized variational principle of Herglotz reduces to the classical variational principle Theorem 5 reduces to the classical second Noether theorem. The proof of this theorem can be found in Georgieva *et al* [4]. The interested reader can find a generalization of the variational principle of Herglotz in the case of several independent variables in Georgieva *et al* [3].

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