

MOTION OF CHARGED PARTICLES IN TWO-STEP NILPOTENT LIE GROUPS*

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Abstract. We formulate the equation of motion of a charged particle in a Riemannian manifold with a closed two form. Since a two-step nilpotent Lie group has natural left-invariant closed two forms, it is natural to consider the motion of a charged particle in a simply connected two-step nilpotent Lie groups with a left invariant metric. We study the behavior of the motion of a charged particle in the above spaces.

1. Introduction

Let Ω be a closed two-form on a connected Riemannian manifold $(M, \langle \cdot, \cdot \rangle)$, where $\langle \cdot, \cdot \rangle$ is a Riemannian metric on M . We denote by $\Lambda^m(M)$ the space of m -forms on M . We denote by $\iota(X) : \Lambda^m(M) \rightarrow \Lambda^{m-1}(M)$ the interior product operator induced from a vector field X on M , and by $\mathcal{L} : T(M) \rightarrow T^*(M)$, the Legendre transformation from the tangent bundle $T(M)$ over M onto the cotangent bundle $T^*(M)$ over M , which is defined by

$$\mathcal{L} : T(M) \rightarrow T^*(M), \quad u \mapsto \mathcal{L}(u), \quad \mathcal{L}(u)(v) = \langle u, v \rangle, \quad u, v \in T(M). \quad (1)$$

A curve $\chi(t)$ in M is referred as a *motion of a charged particle under electromagnetic field* Ω , if it satisfies the following second order differential equation

$$\nabla_{\dot{\chi}} \ddot{\chi} = \mathcal{L}^{-1}(\iota(\dot{\chi})\Omega) \quad (2)$$

where ∇ is the Levi-Civita connection of M . Here $\nabla_{\dot{\chi}} \ddot{\chi}$ means the acceleration of the charged particle. Since $\mathcal{L}^{-1}(\iota(\dot{\chi})\Omega)$ is perpendicular to the direction $\dot{\chi}$ of the movement, $\mathcal{L}^{-1}(\iota(\dot{\chi})\Omega)$ means the **Lorentz force**. The speed $|\dot{\chi}|$ is a conservative constant for a charged particle. When $\Omega = 0$, then the motion of a

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charged particle is nothing but a geodesic. The equation (2) originated in the theory of relativity (see [2] for details).

In this paper, we deal with the motion of a charged particles in a simply connected two-step nilpotent Lie group N with a left invariant Riemannian metric. Since a two-step nilpotent Lie group has a non-trivial center Z , we can construct a left-invariant closed two form $\Omega_{\mathfrak{a}_0}$ from an element $\mathfrak{a}_0 \in Z$ specified below and consider the motion of a charged particle under the electromagnetic field $\Omega_{\mathfrak{a}_0}$. H. Naitoh and Y. Sakane proved that there are no closed geodesics in a simply connected nilpotent Lie group. In contrast with geodesics, there exist motions of charged particles which are periodic. Kaplan defined a H-type Lie group, which is a kind of two-step nilpotent Lie groups. We study the motion of a charged particle in a H-type Lie group more explicitly than in a general two-step nilpotent Lie group.

2. Charged Particles in Two-step Nilpotent Lie Groups

Let N be a simply connected two-step nilpotent Lie group with a left-invariant Riemannian metric $\langle \cdot, \cdot \rangle$. Denote by \mathfrak{g} the vector space consisting of all left-invariant vector fields on N . Since N is two-step nilpotent, \mathfrak{g} has a non-trivial center \mathfrak{z} . Let $\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{h}$ be an orthogonal direct sum decomposition of \mathfrak{g} , then $\mathfrak{z}, \mathfrak{h} \subset \mathfrak{g}$. For $\mathfrak{a}_0 \in \mathfrak{z}$, we define a linear transformation $\phi_{\mathfrak{a}_0}$ on \mathfrak{h} by

$$\langle \phi_{\mathfrak{a}_0}(X), Y \rangle = \langle \mathfrak{a}_0, X, Y \rangle, \quad X, Y \in \mathfrak{h}.$$

We extend $\phi_{\mathfrak{a}_0}$ to a linear transformation on \mathfrak{g} by $\phi = 0$ on \mathfrak{z} , which is also denoted by $\phi_{\mathfrak{a}_0}$. We can regard $\phi_{\mathfrak{a}_0}$ as a left-invariant $(1, 1)$ -tensor on N . Then $\phi_{\mathfrak{a}_0}$ is skew-symmetric with respect to the left-invariant Riemannian metric $\langle \cdot, \cdot \rangle$ since

$$\langle \phi_{\mathfrak{a}_0}(X), Y \rangle + \langle X, \phi_{\mathfrak{a}_0}(Y) \rangle = \langle \mathfrak{a}_0, X, Y \rangle + \langle \mathfrak{a}_0, Y, X \rangle = 0$$

for any left invariant vector fields $X, Y \in \mathfrak{g}$. If we define a left-invariant two-form $\Omega_{\mathfrak{a}_0}$ on N by

$$\Omega_{\mathfrak{a}_0}(X, Y) = \langle X, \phi_{\mathfrak{a}_0}(Y) \rangle, \quad X, Y \in \mathfrak{g}$$

then a simple calculation implies that $\Omega_{\mathfrak{a}_0}$ is closed. In fact, for any X_1, X_2 and X_3 in \mathfrak{g} we have

$$\begin{aligned} 3!(d\Omega_{\mathfrak{a}_0})(X_1, X_2, X_3) &= \Omega_{\mathfrak{a}_0}(X_1, X_2, X_3) \\ &= \langle X_1, X_2, \phi_{\mathfrak{a}_0}(X_3) \rangle = 0 \end{aligned}$$

where we denote by $\langle X_1, X_2, X_3 \rangle$ the cyclic sum, and the last equality follows from the fact that $X_1, X_2 \in \mathfrak{h}$ and $\phi(X_3) \in \mathfrak{z}$. The equation of motion of the charged particle under the electromagnetic field $\Omega_{\mathfrak{a}_0}$ is

$$\nabla_{\dot{x}} \dot{x} = \phi_{\mathfrak{a}_0}(\dot{x}). \quad (3)$$

Here a curve in a manifold is **simple** if it is a simply closed periodic curve, or if it does not intersect itself. Since N is simply connected, the one dimensional de-Rham cohomology group vanishes. Hence we get the following theorem using Theorem 9 in [2].

Theorem 1. *The motion of a charged particle (3) in a simply connected two-step nilpotent Lie group is simple.*

Now we will construct explicitly a simply connected two step nilpotent Lie group with a left-invariant Riemannian metric from an (abstract) two-step nilpotent Lie algebra with an inner product \langle , \rangle . In order to do this, we recall a **Hausdorff formula** for a Lie group (see [1, p. 106]), which states that

$$\exp X \exp Y = \exp \left(X + Y + \frac{1}{2} [X, Y] + \dots \right).$$

If the Lie group is two-step nilpotent, then the higher terms $+\dots$ on the right hand side vanish. Based on the Hausdorff formula, we define a Lie group structure on itself by

$$X \cdot Y = X + Y + \frac{1}{2} [X, Y], \quad X, Y \in \mathfrak{g}.$$

The identity element in this group is 0 , and the inverse element of $x \in \mathfrak{g}$ is equal to $-x$. We denote by $N = (\mathfrak{g}, \cdot)$ the so obtained Lie group. The center of N coincides with \mathfrak{z} . Denote by $\text{Lie}(N)$ the Lie algebra consisting of all left-invariant vector fields on N . Then $\text{Lie}(N)$ is identified with \mathfrak{g} as a Lie algebra as mentioned below. Since N is a Euclidean space as a manifold, we can identify $T_0(N)$ with \mathfrak{g} as vector spaces. The identification induces a Lie algebra structure on $T_0(N)$. For $X \in T_0(N)$, we denote by $\tilde{X} \in \text{Lie}(N)$ the left-invariant vector field on N such that $\tilde{X}_0 = X$. The mapping defined by $\mathfrak{g} = T_0(N) \rightarrow \text{Lie}(N), X \mapsto \tilde{X}$ gives an isomorphism as Lie algebras. Hence N is a simply connected two-step nilpotent Lie group whose Lie algebra is \mathfrak{g} . The inner product \langle , \rangle on \mathfrak{g} induces a left-invariant Riemannian metric \langle , \rangle on N . Using this notation, we have

$$\Omega_{a_0}(\tilde{X}, \tilde{Y}) = \langle \tilde{X}, \phi \tilde{Y} \rangle = \langle \tilde{a}_0, \tilde{Y}, \tilde{X} \rangle = \langle a_0, Y, X \rangle.$$

The exponential mapping $\exp : \mathfrak{g} \rightarrow N$ is equal to identity mapping. Hence for $X \in T_0(N)$, we have

$$\tilde{X}_x = \frac{d}{dt}(x \cdot tX)|_{t=0} = \frac{d}{dt} \left(x + tX + \frac{t}{2} [x, X] \right)_{|t=0} \in T_x(N).$$

Since the Riemannian metric on N is left-invariant, the left action of N on N itself is isometric. Hence $X \in T_0(N)$ induces a Killing vector field X^* on N by

$$X^*_x = \frac{d}{dt}(\exp tX)x|_{t=0} = \frac{d}{dt}(tX + x + \frac{t}{2} [X, x])|_{t=0} \in T_x(N).$$

The Killing vector field X^* is right-invariant.

Lemma 1. *The mapping defined by*

$$\rightarrow , \quad X \mapsto X + \frac{1}{2} X, x$$

is a linear isomorphism.

Proof: Since the mapping is clearly linear, it is sufficient to prove that it is injective. In order to do this, we study the kernel of the mapping. Suppose that $X \in$ satisfy the condition $X + \frac{1}{2} X, x = 0$. Decompose X as $X = X_1 + X_2$ where $X_1 \in \perp$ and $X_2 \in$, then $X_1 + (X_2 + \frac{1}{2} X_1, x) = 0$. This implies $X_1 = 0$ and $X_2 + \frac{1}{2} X_1, x = 0$. Hence we have $X_2 = 0$, hence, $X = 0$. \square

By the lemma above, we have $T_x(N) = \text{span}\{X_x^* ; X \in \}$ for any x in N . The Killing vector field X^* is an infinitesimal automorphism of ϕ .

Lemma 2. *Let X be in $T_0(N) =$. For a fixed $x \in N$, we have $X_x^* = \tilde{W}_x$, where we set $W = X + X, x$.*

Proof: Since

$$\begin{aligned} \tilde{W}_x &= \frac{d}{dt} \left(x + tX + t X, x + \frac{t}{2} x, X + X, x \right) \Big|_{t=0} \\ &= \frac{d}{dt} \left(x + tX + \frac{t}{2} X, x \right) \Big|_{t=0} = X_x^* \end{aligned}$$

we have the assertion. \square

Lemma 3. *Define a one-form η_{a_0} on N by*

$$\eta_{a_0}(X_x^*) = \langle x, X, a_0 \rangle, \quad X \in .$$

Then $\iota(X^)\Omega_{a_0} = d(\eta_{a_0}(X^*))$ for any X in .*

Proof: Let X and Y be in . By Lemma 2, we have

$$\begin{aligned} (\iota(X_x^*)\Omega_{a_0})(\tilde{Y}_x) &= \Omega_{a_0}(X_x^*, \tilde{Y}_x) \\ &= \Omega_{a_0}(\tilde{W}_x, \tilde{Y}_x) \\ &= \Omega_{a_0}(X + X, x), Y) \\ &= \langle a_0, Y, X + X, x \rangle = \langle a_0, Y, X \rangle. \end{aligned}$$

Using the above equation, we have also

$$\begin{aligned} d(\eta_{\mathfrak{a}_0}(X^*))(\tilde{Y}_x) &= \tilde{Y}_x(\eta_{\mathfrak{a}_0}(X^*)) \\ &= \frac{d}{dt} \eta_{\mathfrak{a}_0}(X_{x+tY+\frac{t}{2}[X,Y]}^*)|_{t=0} \\ &= \frac{d}{dt} \langle x+tY+\frac{t}{2}x, Y, X, \mathfrak{a}_0 \rangle \\ &= \langle Y, X, \mathfrak{a}_0 \rangle = (\iota(X^*)\Omega_{\mathfrak{a}_0})(\tilde{Y}_x). \end{aligned}$$

Hence we get $d(\eta_{\mathfrak{a}_0}(X^*)) = \iota(X^*)\Omega_{\mathfrak{a}_0}$. □

Denote by $T_x(N) \rightarrow T_0(N); v \mapsto v_n$ the usual parallel translation in the Euclidean space : Take a curve $c(t)$ in N such that $c(0) = x, \dot{c}(0) = v$. Then $v_n = \frac{d}{dt}(c(t) - x)|_{t=0}$. The following lemma gives a relation between the two linear isomorphisms $L_x^{-1} : T_x(N) \rightarrow T_0(N)$ and $T_x(N) \rightarrow T_0(N), v \mapsto v_n$.

Lemma 4. $L_x^{-1}v = v_n - \frac{1}{2}x, v_n$ for $v \in T_x(N)$.

Proof: Take a curve $c(t)$ in N such that $c(0) = x, \dot{c}(0) = v$. Then

$$\begin{aligned} L_x^{-1}v &= L_{-x}v = \frac{d}{dt} \left(x + c(t) - \frac{1}{2}x, c(t) \right) |_{t=0} \\ &= \frac{d}{dt} \left(c(t) - x - \frac{1}{2}x, c(t) - x \right) |_{t=0} = v_n - \frac{1}{2}x, v_n. \end{aligned}$$

Hence we have the assertion. □

Similarly we define $T_y(\mathbb{R}^\perp) \rightarrow T_0(\mathbb{R}^\perp), u \mapsto u_{\mathfrak{J}^\perp}$ and $T_z(\mathbb{R}^\perp) \rightarrow T_0(\mathbb{R}^\perp), w \mapsto w_{\mathfrak{J}}$. Since \mathfrak{g} is abelian, we have $L_z^{-1}w = w_{\mathfrak{J}}$ for $w \in T_z(\mathbb{R}^\perp)$. Hence we can write $w = w_{\mathfrak{J}}$. Let $x \in \mathbb{R}^\perp$ and $v \in T_x(\mathbb{R}^\perp)$. Expressing x and v as $x = y + z$ and $v = v_1 + v_2$, where $y \in \mathbb{R}^\perp, z \in \mathbb{R}^\perp, v_1 \in T_y(\mathbb{R}^\perp)$ and $v_2 \in T_z(\mathbb{R}^\perp)$ we obtain

$$L_x^{-1}v = (v_1)_{\mathfrak{J}^\perp} + \left(v_2 - \frac{1}{2}y, (v_1)_{\mathfrak{J}^\perp} \right). \tag{4}$$

Proposition 1. Let $x(t) = y(t) + z(t)$ be a curve in \mathbb{R}^\perp , where $y(t) \in \mathbb{R}^\perp$ and $z(t) \in \mathbb{R}^\perp$. Assume that $y(0) = 0$. Then $x(t)$ describes the motion of a charged particle (3) if and only if

$$\dot{y}(t)_{\mathfrak{J}^\perp} - \Phi_{z(0)+\mathfrak{a}_0}y(t) = \dot{y}(0), \quad \dot{z}(t) - \frac{1}{2}y(t), \dot{y}(t)_{\mathfrak{J}^\perp} = \dot{z}(0). \tag{5}$$

Proof: Taking the inner product of (3) and the Killing vector field X^* for $X \in \mathfrak{g}$, we have

$$\frac{d}{dt} \langle \dot{x}, X^* \rangle = \Omega(X^*, \dot{x}) = (\iota(X^*)\Omega)(\dot{x}).$$

Using Lemma 3 we find

$$\frac{d}{dt} \langle \dot{x}, X^* \rangle = (d(\eta(X^*))) (\dot{x}) = \frac{d}{dt} \eta(X_{x(t)}^*).$$

Since $T_x(N) = \text{span}\{X_x^*; X \in \mathfrak{g}\}$, the equation (3) is equivalent to

$$\frac{d}{dt} (\langle \dot{x}(t), X_{x(t)}^* \rangle - \eta(X_{x(t)}^*)) = 0.$$

By the definition of η , we have

$$\eta(X_{x(t)}^*) = \langle x(t), X, \alpha_0 \rangle = \langle \phi_{\alpha_0}(y(t)), X \rangle.$$

Since $\langle \cdot, \cdot \rangle$ is left invariant

$$\begin{aligned} \langle \dot{x}, X_{x(t)}^* \rangle &= \langle L_x^{-1} \dot{x}, L_x^{-1} X_x^* \rangle \\ &= \left\langle \dot{y}_{3^\perp} + \left(\dot{z} - \frac{1}{2} y, \dot{y}_{3^\perp} \right), X + X, x \right\rangle \\ &= \langle \dot{y}_{3^\perp}, X \rangle + \left\langle \dot{z} - \frac{1}{2} y, \dot{y}_{3^\perp}, X + X, x \right\rangle \end{aligned}$$

where we have used Lemma 2 and equation (4). Hence the equation (3) is equivalent to

$$\frac{d}{dt} \left(\langle \dot{y}_{3^\perp} - \phi_{\alpha_0}(y), X \rangle + \left\langle \dot{z} - \frac{1}{2} y, \dot{y}_{3^\perp}, X + X, y \right\rangle \right) = 0.$$

Taking $X \in \mathfrak{g}$, we have

$$\dot{z}(t) - \frac{1}{2} y(t), \dot{y}(t)_{3^\perp} = \dot{z}(0)$$

where we have used the initial condition $y(0) = 0$. Next, taking $X \in \mathfrak{g}^\perp$, we have

$$\frac{d}{dt} \left(\langle \dot{y}_{3^\perp} - \phi_{\alpha_0}(y), X \rangle + \langle \dot{z}(0), X, y \rangle \right) = 0.$$

Taking into account the initial condition $y(0) = 0$, we finally have

$$\dot{y}(t)_{3^\perp} - \phi_{z(0)+\alpha_0} y(t) = \dot{y}(0).$$

□

Proposition 2. The motion of a charged particle (3) with $y(0) = 0$ is given by the equations

$$\begin{aligned} y(t) &= \exp t\phi_{z(0)+\alpha_0} \int_0^t \exp(-s\phi_{z(0)+\alpha_0}) \dot{y}(0) ds \\ z(t) &= z(0) + tz(0) + \frac{1}{2} \int_0^t y(s), (\exp s\phi_{z(0)+\alpha_0}) \dot{y}(0) ds. \end{aligned}$$

Proof: Using the first equation of (5) with $y(0) = 0$, we have

$$y(t) = \exp t\phi_{\dot{z}(0)+\alpha_0} \int_0^t \exp(-t\phi_{\dot{z}(0)+\alpha_0}) \dot{y}(0) dt.$$

Hence

$$\phi_{\dot{z}(0)+\alpha_0} y(t) = (\exp t\phi_{\dot{z}(0)+\alpha_0} - 1) \dot{y}(0)$$

which implies that

$$\phi_{\dot{z}(0)+\alpha_0} y(t) + \dot{y}(0) = (\exp t\phi_{\dot{z}(0)+\alpha_0}) \dot{y}(0).$$

Using the second and the first equation from (5)

$$\begin{aligned} z(t) &= z(0) + t\dot{z}(0) + \frac{1}{2} \int_0^t y(t), \dot{y}(t)_{3^\perp} dt \\ &= z(0) + t\dot{z}(0) + \frac{1}{2} \int_0^t y(t), \phi_{\dot{z}(0)+\alpha_0} y(t) + \dot{y}(0) dt \\ &= z(0) + t\dot{z}(0) + \frac{1}{2} \int_0^t y(t), (\exp t\phi_{\dot{z}(0)+\alpha_0}) \dot{y}(0) dt. \end{aligned}$$

Hence we get the assertion. \square

When $\phi_{\dot{z}(0)+\alpha_0} = 0$, then, using the above Proposition, we get $y(t) = t\dot{y}(0)$ and

$$z(t) = z(0) + t\dot{z}(0) + \frac{1}{2} \int_0^t t\dot{y}(0), \dot{y}(0) dt = z(0) + t\dot{z}(0).$$

Lemma 5. *The equation of motion (3) implies the following relation*

$$\frac{d}{dt} (\langle z(t), \dot{z}(0) + \alpha_0 \rangle + \frac{1}{2} \langle y(t), \dot{y}(0) \rangle) = |\dot{z}(0)|^2 + \langle \dot{z}(0), \alpha_0 \rangle + \frac{1}{2} |\dot{y}_{3^\perp}|^2.$$

Proof: Taking the inner product of the second equation of (5) with $\dot{z}(0) + \alpha_0$, we have

$$\langle \dot{z}, \dot{z}(0) + \alpha_0 \rangle - \frac{1}{2} \langle y, \dot{y}_{3^\perp}, \dot{z}(0) + \alpha_0 \rangle = |\dot{z}(0)|^2 + \langle \dot{z}(0), \alpha_0 \rangle.$$

Using equation (5) again produces

$$\begin{aligned} \langle y, \dot{y}_{3^\perp}, \dot{z}(0) + \alpha_0 \rangle &= \langle \phi_{\dot{z}(0)+\alpha_0} y, \dot{y}_{3^\perp} \rangle \\ &= \langle \dot{y}_{3^\perp} - \dot{y}(0), \dot{y}_{3^\perp} \rangle \\ &= |\dot{y}_{3^\perp}|^2 - \langle \dot{y}_{3^\perp}, \dot{y}(0) \rangle = |\dot{y}_{3^\perp}|^2 - \frac{d}{dt} \langle y(t), \dot{y}(0) \rangle. \end{aligned}$$

Hence

$$\frac{d}{dt} (\langle z(t), \dot{z}(0) + \alpha_0 \rangle + \frac{1}{2} \langle y(t), \dot{y}(0) \rangle) = |\dot{z}(0)|^2 + \langle \dot{z}(0), \alpha_0 \rangle + \frac{1}{2} |\dot{y}_{3^\perp}|^2.$$

\square

Applying the lemma above for geodesics, we can re-demonstrate the following theorem of Naitoh-Sakane.

Theorem 2. (Naitoh-Sakane [4, Corrolary 3.3]) *Every geodesic in any simply connected two-step nilpotent Lie group with a left-invariant Riemannian metric does not intersect itself.*

Proof: Let $x(t) = y(t) + z(t) \in \mathbb{N}$ be a geodesic with $y(0) = 0$. Applying Lemma 5 with $a_0 = 0$

$$\frac{d}{dt} \left(\langle z(t), \dot{z}(0) \rangle + \frac{1}{2} \langle y(t), \dot{y}(0) \rangle \right) = |\dot{z}(0)|^2 + \frac{1}{2} |\dot{y}_{3^\perp}|^2 > 0.$$

Hence $\langle z(t), \dot{z}(0) \rangle + \frac{1}{2} \langle y(t), \dot{y}(0) \rangle$ is monotone increasing. Thus $x(t)$ is not periodic. Since we have already proved that $x(t)$ is simple by Theorem 1, we get the assertion. \square

The author thinks that the above proof is easier than the original proof of Naitoh-Sakane.

3. Charged Particles in H-type Lie Groups

In this section, we study the motion of a charged particle in a simply connected H-type Lie group. First we review the definition of H-type Lie algebra according to Kaplan. Let $(U, \langle \cdot, \cdot \rangle)$ and $(V, \langle \cdot, \cdot \rangle)$ be finite-dimensional real vector spaces with inner products $\langle \cdot, \cdot \rangle$. Denote by $\text{End}(V)$ the vector space consisting of all linear transformations on V . We assume that there exists a linear mapping $j : U \rightarrow \text{End}(V)$ such that

$$j(a)^2 = -|a|^2 I, \quad |j(a)x| = |a||x|, \quad a \in U, \quad x \in V. \quad (6)$$

In other words, V is a Clifford module over the Clifford algebra generated by U . By (6) we have

$$\begin{aligned} \langle j(a)x, j(b)x \rangle &= \langle a, b \rangle |x|^2, & \langle j(a)x, j(a)y \rangle &= -|a|^2 \langle x, y \rangle \\ \langle j(a)x, y \rangle + \langle x, j(a)y \rangle &= 0, & a, b \in U, \quad x, y \in V. \end{aligned}$$

Define a bi-linear mapping $\langle \cdot, \cdot \rangle : V \times V \rightarrow U$ via the formula

$$\langle a, x, y \rangle = \langle j(a)x, y \rangle, \quad a \in U, \quad x, y \in V. \quad (7)$$

Then $\langle \cdot, \cdot \rangle$ is alternative. Substituting $j(b)x$ into y , we have

$$\langle a, x, j(b)x \rangle = \langle j(a)x, j(b)x \rangle = \langle a, b \rangle |x|^2.$$

Hence

$$x, j(b)x = |x|^2 b, \quad b \in U, \quad x \in V. \quad (8)$$

We denote by $\mathfrak{g} = \mathfrak{U} \oplus \mathfrak{V}$ the orthogonal direct sum of \mathfrak{U} and \mathfrak{V} , and define a Lie algebra structure on \mathfrak{g} by

$$\alpha + x, \beta + y = x, y \in \mathfrak{U}, \quad \alpha, \beta \in \mathfrak{U}, \quad x, y \in \mathfrak{V}.$$

Then the Lie algebra \mathfrak{g} is called **H-type**. Since the H-type Lie algebra \mathfrak{g} is a kind of two-step nilpotent Lie algebra with an inner product, we can define a Lie group structure on \mathfrak{g} with a left-invariant Riemannian metric, whose Lie algebra is \mathfrak{g} itself as we mentioned in the previous section. For $\alpha_0 \in \mathfrak{U}$, we consider the equation

$$\nabla_x \dot{x} = j(\alpha_0)\dot{x} \quad (9)$$

of motion of a charged particle. If we express its trajectory as $x(t) = y(t) + z(t)$ where $y(t) \in \mathfrak{V}, z(t) \in \mathfrak{U}$, then (9) is equivalent to

$$\dot{y}(t)_V = j(\dot{z}(0) + \alpha_0)y(t) = \dot{y}(0) \quad (10)$$

where $T_y(V) \rightarrow V, w \mapsto w_V$ denotes the usual parallel translation in V . Here we have used equation (5).

Theorem 3. *Let $x(t) = y(t) + z(t) \in \mathfrak{N}$ (where $y(t) \in \mathfrak{V}, z(t) \in \mathfrak{U}$) is a motion of a charged particle (9) with $x(0) = 0$.*

1) *When $\dot{z}(0) + \alpha_0 = 0$, then $x(t) = t\dot{x}(0)$.*

2) *When $\dot{z}(0) + \alpha_0 \neq 0$, then*

$$y(t) = \frac{\sin(t|\dot{z}(0) + \alpha_0|)}{|\dot{z}(0) + \alpha_0|} \dot{y}(0) + \frac{1 - \cos(t|\dot{z}(0) + \alpha_0|)}{|\dot{z}(0) + \alpha_0|^2} j(\dot{z}(0) + \alpha_0)\dot{y}(0)$$

$$z(t) = t\dot{z}(0) + \frac{t|\dot{y}(0)|^2}{2|\dot{z}(0) + \alpha_0|^2} (\dot{z}(0) + \alpha_0) - \frac{\sin(t|\dot{z}(0) + \alpha_0|)}{2|\dot{z}(0) + \alpha_0|^3} |\dot{y}(0)|^2 (\dot{z}(0) + \alpha_0).$$

The curve $y(t)$ is a circle in V . The motion of a charged particle is periodic if and only if

$$\alpha_0 = \left(\frac{|\dot{y}(0)|^2}{2|\dot{z}(0)|^2} + 1 \right) \dot{z}(0).$$

In this case $x(t)$ is an elliptic motion.

Remark 1. When $x(t)$ is a geodesic, the condition $\alpha_0 = 0$ implies the theorem of Kaplan [3].

Proof: 1) is clear from (10). We will show 2). Using the first equation of (10), we have

$$y(t) = \frac{\sin(t|\dot{z}(0) + \alpha_0|)}{|\dot{z}(0) + \alpha_0|} \dot{y}(0) + \frac{1 - \cos(t|\dot{z}(0) + \alpha_0|)}{|\dot{z}(0) + \alpha_0|^2} j(\dot{z}(0) + \alpha_0)\dot{y}(0)$$

which implies that

$$\dot{y}(t)_V = \cos(t|\dot{z}(0) + \alpha_0|)\dot{y}(0) + \frac{\sin(t|\dot{z}(0) + \alpha_0|)}{|\dot{z}(0) + \alpha_0|} j(\dot{z}(0) + \alpha_0)\dot{y}(0).$$

Using the equation above, we have

$$\mathbf{y}(t)_{\mathbf{V}}, \dot{\mathbf{y}}(t) = \frac{1}{|\dot{\mathbf{z}}(0) + \mathbf{a}_0|^2} \cos(t|\dot{\mathbf{z}}(0) + \mathbf{a}_0|) \dot{\mathbf{y}}(0), j(\dot{\mathbf{z}}(0) + \mathbf{a}_0)\dot{\mathbf{y}}(0).$$

Further the second equation of (10) gives

$$\begin{aligned} \dot{\mathbf{z}}(t) &= \dot{\mathbf{z}}(0) + \frac{1}{2|\dot{\mathbf{z}}(0) + \mathbf{a}_0|^2} \cos(t|\dot{\mathbf{z}}(0) + \mathbf{a}_0|) \dot{\mathbf{y}}(0), j(\dot{\mathbf{z}}(0) + \mathbf{a}_0)\dot{\mathbf{y}}(0) \\ &= \dot{\mathbf{z}}(0) + \frac{1}{2|\dot{\mathbf{z}}(0) + \mathbf{a}_0|^2} (\dot{\mathbf{z}}(0) + \mathbf{a}_0) |\dot{\mathbf{y}}(0)|^2 \end{aligned} \quad (11)$$

where we have used the equation (8). Since

$$\begin{aligned} \mathbf{y}(t) &= \frac{1}{|\dot{\mathbf{z}}(0) + \mathbf{a}_0|} j \left(\frac{\dot{\mathbf{z}}(0) + \mathbf{a}_0}{|\dot{\mathbf{z}}(0) + \mathbf{a}_0|} \right) \dot{\mathbf{y}}(0) = \frac{\sin(|\dot{\mathbf{z}}(0) + \mathbf{a}_0|t)}{|\dot{\mathbf{z}}(0) + \mathbf{a}_0|} \dot{\mathbf{y}}(0) \\ &\quad + \frac{\cos(|\dot{\mathbf{z}}(0) + \mathbf{a}_0|t)}{|\dot{\mathbf{z}}(0) + \mathbf{a}_0|} j \left(\frac{\dot{\mathbf{z}}(0) + \mathbf{a}_0}{|\dot{\mathbf{z}}(0) + \mathbf{a}_0|} \right) \dot{\mathbf{y}}(0) \end{aligned}$$

the curve $\mathbf{y}(t)$ is a circle in \mathbf{V} whose center is $\frac{1}{|\dot{\mathbf{z}}(0) + \mathbf{a}_0|} j \left(\frac{\dot{\mathbf{z}}(0) + \mathbf{a}_0}{|\dot{\mathbf{z}}(0) + \mathbf{a}_0|} \right) \dot{\mathbf{y}}(0)$ and the radius is $\frac{|\dot{\mathbf{y}}(0)|}{|\dot{\mathbf{z}}(0) + \mathbf{a}_0|}$. The periodic condition is as follows

$$\begin{aligned} \mathbf{x}(t) \text{ is periodic} &\Leftrightarrow \dot{\mathbf{z}}(0) + \frac{|\dot{\mathbf{y}}(0)|^2}{2|\dot{\mathbf{z}}(0) + \mathbf{a}_0|^2} (\dot{\mathbf{z}}(0) + \mathbf{a}_0) = 0 \\ &\Leftrightarrow \mathbf{a}_0 = \left(\frac{|\dot{\mathbf{y}}(0)|^2}{2|\dot{\mathbf{z}}(0)|^2} + 1 \right) \dot{\mathbf{z}}(0). \end{aligned}$$

In this case, since

$$\begin{aligned} \mathbf{x}(t) + \frac{2|\dot{\mathbf{z}}(0)|}{|\dot{\mathbf{y}}(0)|^2} j \left(\frac{\dot{\mathbf{z}}(0)}{|\dot{\mathbf{z}}(0)|} \right) \dot{\mathbf{y}}(0) &= \frac{2|\dot{\mathbf{z}}(0)|}{|\dot{\mathbf{y}}(0)|^2} \left(\sin \left(\frac{|\dot{\mathbf{y}}(0)|^2}{2|\dot{\mathbf{z}}(0)|} t \right) (\dot{\mathbf{y}}(0) + \dot{\mathbf{z}}(0)) \right. \\ &\quad \left. + \cos \left(\frac{|\dot{\mathbf{y}}(0)|^2}{2|\dot{\mathbf{z}}(0)|} t \right) j \left(\frac{\dot{\mathbf{z}}(0)}{|\dot{\mathbf{z}}(0)|} \right) \dot{\mathbf{y}}(0) \right) \end{aligned}$$

the curve $\mathbf{x}(t)$ is an elliptic such that the ratio of the long axis to the short axis is equal to $\sqrt{|\dot{\mathbf{y}}(0)|^2 + |\dot{\mathbf{z}}(0)|^2} / |\dot{\mathbf{y}}(0)|$. \square

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