

EINSTEIN METRICS WITH TWO-DIMENSIONAL KILLING LEAVES AND THEIR APPLICATIONS IN PHYSICS

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Abstract. Solutions of vacuum Einstein’s field equations, for the class of pseudo-Riemannian four-metrics admitting a non Abelian two dimensional Lie algebra of Killing fields, are explicitly described. When the distribution orthogonal to the orbits is completely integrable and the metric is not degenerate along the orbits, these solutions are parameterized either by solutions of a transcendental equation (the tortoise equation), or by solutions of a linear second order differential equation in two independent variables. Metrics, corresponding to solutions of the tortoise equation, are characterized as those that admit a three dimensional Lie algebra of Killing fields with two dimensional leaves. Metrics, corresponding to the case in which the commutator of the two Killing fields is isotropic, represent nonlinear gravitational waves.

1. Introduction

The aim of this paper is to illustrate some interesting and, in some sense, surprising physical properties of special solutions of Einstein field equations belonging to the larger class of Einstein metrics invariant for a non-Abelian Lie algebra of Killing vector fields generating a two dimensional distribution.

Some decades ago, by using a suitable generalization of the *Inverse Scattering Transform*, Belinsky and Sakharov [3] were able to determine four-dimensional Ricci-flat Lorentzian metrics invariant for an Abelian two dimensional Lie algebra of Killing vector fields such that the distribution \mathcal{D}^\perp orthogonal to the one, say \mathcal{D} , generated by the Killing fields is transversal to \mathcal{D} and Frobenius-integrable.

Thus, as a first step, it has been natural to consider [16] the problem of characterizing all gravitational fields g admitting a Lie algebra \mathcal{G} of Killing fields such that

- I* the distribution \mathcal{D} , generated by vector fields of \mathcal{G} , is two dimensional
II the distribution \mathcal{D}^\perp , orthogonal to \mathcal{D} is integrable and transversal to \mathcal{D} .

As we will see in Sections 4 and 5, the condition of transversality can be relaxed. This case, when the metric g restricted to any integral (two dimensional) submanifold (Killing leaf) of the distribution \mathcal{D} is degenerate, splits naturally into two sub-cases according to whether the rank of g restricted to Killing leaves is 1 or 0. Sometimes, in order to distinguish various cases occurring in the sequel, the notation (\mathcal{G}, r) will be used: metrics satisfying the conditions *I* and *II* will be called of $(\mathcal{G}, 2)$ -type, metrics satisfying conditions *I* and *II*, except the transversality condition, will be called of $(\mathcal{G}, 0)$ -type or of $(\mathcal{G}, 1)$ -type according to the rank of their restriction to Killing leaves.

According to whether the dimension of \mathcal{G} is three or two, two qualitatively different cases can occur. Both of them, however, have in common the important feature that all manifolds satisfying the assumptions *I* and *II* are in a sense fibered over ζ -complex curves [18].

When $\dim \mathcal{G} = 3$, assumption *II* follows from *I* and the local structure of this class of Einstein metrics can be explicitly described. Some well known exact solutions, e.g. Schwarzschild, belong to this class.

A two dimensional \mathcal{G} , is either Abelian (\mathcal{A}_2) or non-Abelian (\mathcal{G}_2) and a metric g satisfying *I* and *II*, with $\mathcal{G} = \mathcal{A}_2$ or \mathcal{G}_2 , will be called \mathcal{G} -integrable. The study of \mathcal{A}_2 -integrable Einstein metrics goes back to Einstein and Rosen [9]. Recent results can be found in [7].

The greater rigidity of \mathcal{G}_2 -integrable metrics, for which some partial results can be found in [1, 8, 10], allows an exhaustive analysis. It will be shown that the ones of $(\mathcal{G}, 2)$ -type are parameterized by solutions of a linear second order differential equation on the plane which, in its turn, depends linearly on the choice of a ζ -harmonic function (see later). Thus, this class of solutions has a *bilinear structure* and, as such, admits two *superposition laws*.

All possible situations, corresponding to a two dimensional Lie algebras of isometries, are described in Table 1 where a non integrable two dimensional distribution which is part of a three dimensional integrable distribution has been called *semi-integrable* and in which the cases indicated with bold letters have been essentially solved [2, 7, 16–18].

In Section 1, four dimensional metrics of $(\mathcal{G}_2, 2)$ -type invariant for a non Abelian two dimensional Lie algebra are characterized from a geometric point of view. The solutions of corresponding Einstein field equations are explicitly written. The construction of global solutions is described in Section 2 and some examples are given in Section 3. Sections 4 and 5 are devoted to metrics of $(\mathcal{G}, 1)$ -type and of $(\mathcal{G}, 0)$ -type respectively. In Section 6 the case in which the commutator of

Table 1. Cases indicated with bold characters admit Ricci flat metrics, the remaining ones are under investigation.

	$\mathcal{D}^\perp, r = 0$	$\mathcal{D}^\perp, r = 1$	$\mathcal{D}^\perp, r = 2$
\mathcal{G}_2	integrable	integrable	integrable
\mathcal{G}_2	semi-integrable	semi-integrable	semi-integrable
\mathcal{G}_2	non-integrable	non-integrable	non-integrable
\mathcal{A}_2	integrable	integrable	integrable
\mathcal{A}_2	semi-integrable	semi-integrable	semi-integrable
\mathcal{A}_2	non-integrable	non-integrable	non-integrable

generators of the Lie algebra is of *light-type* is analyzed from a physical point of view. Harmonic coordinates are also introduced. Moreover, the wave-like character of the solutions is checked through the Zel'manov and the Pirani criteria. The canonical Landau-Lifchitz and the Bel energy-momentum pseudo-tensors are introduced and a comparison with the linearised theory is performed. Realistic sources for such gravitational waves are also described. Eventually, the analysis of the polarization leads to the conclusion that these fields are spin-1 gravitational waves.

2. Metrics of $(\mathcal{G}_2, 2)$ - Type

In the following, we will consider four-dimensional manifolds and Greek letters take values from 1 to 4, the first Latin letters take values from 3 to 4, while i, j from 1 to 2. Moreover, $Kil(g)$ will denote the Lie algebra of all Killing fields of a metric g while *Killing algebra* will denote a sub-algebra of $Kil(g)$. Moreover, an integral (two dimensional) submanifold of \mathcal{D} will be called a *Killing leaf*, and an integral (two dimensional) submanifold of \mathcal{D}^\perp *orthogonal leaf*.

2.1. Geometric Aspects

- *Semiadapted coordinates.*

Let g be a metric on the space-time \mathcal{M} (a connected smooth manifold) and \mathcal{G}_2 one of its Killing algebras whose generators X, Y satisfy $[X, Y] = sY$, $s = 0, 1$

The Frobenius distribution \mathcal{D} generated by \mathcal{G}_2 is two-dimensional and in the neighborhood of a non singular point a chart (x^1, x^2, x^3, x^4) exists such that

$$X = \frac{\partial}{\partial x^3}, \quad Y = \exp(sx^3) \frac{\partial}{\partial x^4}.$$

From now on such a chart will be called *semiadapted* (to the Killing fields).

- *Invariant metrics*

It can be easily verified [16, 17] that in a semiadapted chart g has the form

$$g = g_{ij} dx^i dx^j + 2 \left(l_i + s m_i x^4 \right) dx^i dx^3 - 2 m_i dx^i dx^4 \\ + \left(s^2 \lambda \left(x^4 \right)^2 - 2 s \mu x^4 + \nu \right) dx^3 dx^3 + 2 \left(\mu - s \lambda x^4 \right) dx^3 dx^4 \\ + \lambda dx^4 dx^4, \quad i = 1, 2; j = 1, 2$$

with g_{ij} , m_i , l_i , λ , μ , ν arbitrary functions of (x^1, x^2) .

- *Killing leaves.*

Condition II allows to construct semi-adapted charts, with new coordinates (x, y, x^3, x^4) , such that the fields $e_1 = \partial/\partial x$, $e_2 = \partial/\partial y$, belong to \mathcal{D}^\perp . In such a chart, called from now on *adapted*, the components l_i 's and m_i 's vanish.

As it has already said, we will call **Killing leaf** an integral (two dimensional) submanifold of \mathcal{D} and **orthogonal leaf** an integral (two dimensional) submanifold of \mathcal{D}^\perp . Since \mathcal{D}^\perp is transversal to \mathcal{D} , the restriction of g to any Killing leaf, S , is non-degenerate. Thus, $(S, g|_S)$ is a homogeneous two dimensional Riemannian manifold. Then, the Gauss curvature $K(S)$ of the Killing leaves is constant (depending on the leaf). In the appropriate chart $(p = x^3|_S, q = x^4|_S)$ one has

$$g|_S = \left(s^2 \tilde{\lambda} q^2 - 2 s \tilde{\mu} q + \tilde{\nu} \right) dp^2 + 2 \left(\tilde{\mu} - s \tilde{\lambda} q \right) dp dq + \tilde{\lambda} dq^2$$

where $\tilde{\lambda}$, $\tilde{\mu}$, $\tilde{\nu}$, being the restrictions to S of λ , μ , ν , are constants, and

$$K(S) = \tilde{\lambda} s^2 \left(\tilde{\mu}^2 - \tilde{\lambda} \tilde{\nu} \right)^{-1}.$$

2.2. Einstein Metrics When $g(Y, Y) \neq 0$

In the considered class of metrics, vacuum Einstein equations, $R_{\mu\nu} = 0$, can be completely solved [16]. If the Killing field Y is not of *light type*, i.e., $g(Y, Y) \neq 0$, then in the adapted coordinates (x, y, p, q) the general solution is

$$g = f(dx^2 \pm dy^2) + \beta^2 [(s^2 k^2 q^2 - 2slq + m) dp^2 + 2(l - skq) dp dq + k dq^2] \quad (1)$$

where $f = -\Delta_\pm \beta^2 / 2s^2 k$, and $\beta(x, y)$ is a solution of the *tortoise equation*

$$\beta + A \ln |\beta - A| = u(x, y)$$

where A is a constant and the function u is a solution either of the Laplace or the d'Alembert equation, $\Delta_\pm u = 0$, $\Delta_\pm = \partial_{xx}^2 \pm \partial_{yy}^2$, such that $(\partial_x u)^2 \pm (\partial_y u)^2 \neq 0$. The constants k, l, m are constrained by $km - l^2 = \mp 1$, $k \neq 0$ for Lorentzian metrics or by $km - l^2 = \pm 1$, $k \neq 0$ for Kleinian metrics.

Ricci flat manifolds of Kleinian signature possess a number of interesting geometrical properties and undoubtedly deserve attention in their own right. Some topological aspects of these manifolds were studied for the first time in [12], [13] and then in [11]. In recent years the geometry of these manifolds has seen a revival of interest. In part, this is due to the emergence of some new applications in physics.

2.2.1. Canonical Form of Metrics When $g(Y, Y) \neq 0$

The gauge freedom of the above solution, allowed by the function u , can be locally eliminated by introducing the coordinates (u, v, p, q) , the function $v(x, y)$ being conjugate to $u(x, y)$, i.e. $\Delta_{\pm} v = 0$ and $u_x = v_y, u_y = \mp v_x$. In these coordinates the metric g takes the form

$$g = \frac{\exp \frac{u-\beta}{A}}{2s^2k\beta} (du^2 \pm dv^2) + \beta^2 [(s^2k^2q^2 - 2slq + m)dp^2 + 2(l - skq)dpdq + kdq^2]$$

with $\beta(u)$ a solution of $\beta + A \ln |\beta - A| = u$.

2.2.2. Normal Form of Metrics When $g(Y, Y) \neq 0$

In **geographic coordinates** (ϑ, φ) along Killing leaves one has

$$g|_S = \beta^2 [d\vartheta^2 + \mathcal{F}(\vartheta) d\varphi^2]$$

where $\mathcal{F}(\vartheta)$ is equal either to $\sin h^2 \vartheta$ or $-\cosh^2 \vartheta$, depending on the signature of the metric. Thus, in the **normal coordinates**, $(r = 2s^2k\beta, \tau = v, \vartheta, \varphi)$, the metric takes the form (local ‘‘Birkhoff’s theorem’’)

$$g = \varepsilon_1 \left(\left[1 - \frac{A}{r} \right] d\tau^2 \pm \left[1 - \frac{A}{r} \right]^{-1} dr^2 \right) + \varepsilon_2 r^2 [d\vartheta^2 + \mathcal{F}(\vartheta) d\varphi^2] \quad (2)$$

where $\varepsilon_1 = \pm 1, \varepsilon_2 = \pm 1$.

The geometric reason for this form is that, when $g(Y, Y) \neq 0$, a third Killing field exists which together with X and Y constitute a basis of $\mathfrak{s}(2, 1)$. The larger symmetry implies that the geodesic equations describe a *non-commutatively integrable system* [15], and the corresponding geodesic flow projects on the geodesic flow of the metric restricted to the Killing leaves.

The above local form does not allow, however, to treat properly the singularities appearing inevitably in global solutions. The metrics (1), although they all are locally diffeomorphic to (2), play a relevant role in the construction of new global solutions as described in [17, 18].

2.3. Einstein metrics when $g(Y, Y) = 0$

If the Killing field Y is of **light type**, then the general solution of vacuum Einstein equations, in the adapted coordinates (x, y, p, q) , is given by

$$g = 2f(dx^2 \pm dy^2) + \mu[(w(x, y) - 2sq)dp^2 + 2dpdq] \quad (3)$$

where $\mu = A\Phi + B$ with $A, B \in \mathbb{R}$, Φ is a non constant harmonic function of x and y , $f = (\nabla\Phi)^2 \sqrt{|\mu|}/\mu$, and $w(x, y)$ is solution of the **μ - deformed Laplace equation**

$$\Delta_{\pm}w + (\partial_x \ln |\mu|) \partial_x w \pm (\partial_y \ln |\mu|) \partial_y w = 0$$

where Δ_+ (Δ_-) is the Laplace (respectively d'Alembert) operator in the (x, y) - plane. Metrics (3) are Lorentzian if the orthogonal leaves are conformally Euclidean, *i.e.*, the positive sign is chosen, and Kleinian if not. Only the Lorentzian case will be analyzed and these metrics will be called of $(\mathcal{G}_2, 2)$ - **isotropic type**.

In the particular case $s = 1$, $f = 1/2$ and $\mu = 1$, the above (Lorentzian) metrics are locally diffeomorphic to a subclass of the vacuum Peres solutions [14], that for later purpose we rewrite in the form

$$g = dx^2 \pm dy^2 + 2dudv + 2(\varphi_x dx + \varphi_y dy)du. \quad (4)$$

The correspondence between (3) and (4) depends on the special choice of the function $\varphi(x, y, u)$ (which, in general, is harmonic in x and y arbitrarily dependent on u); in our case

$$x \rightarrow x, \quad y \rightarrow y, \quad u \rightarrow u, \quad v \rightarrow v + \varphi(x, y, u)$$

with $h = \varphi_u$.

In the case $\mu = \text{const}$, the μ -deformed Laplace equation reduces to the Laplace equation. For $\mu = 1$, in the harmonic coordinates system (x, y, z, t) defined [4], for $|z - t| \neq 0$, by

$$\begin{aligned} x &= x \\ y &= y \\ z &= \frac{1}{2} [(2q - w(x, y)) \exp(-p) + \exp(p)] \\ t &= \frac{1}{2} [(2q - w(x, y)) \exp(-p) - \exp(p)] \end{aligned}$$

the Einstein metrics (3) take the particularly simple form

$$g = 2f(dx^2 \pm dy^2) + dz^2 - dt^2 + d(w) d(\ln |z - t|). \quad (5)$$

This shows that, when w is constant, the Einstein metrics given by equation (5) are static and, under the further assumption $\Phi = x\sqrt{2}$, they reduce to the Minkowski one. Moreover, when w is not constant, gravitational fields (5) look like a *disturbance* propagating at light velocity along the z direction on the Killing leaves

(integral two dimensional submanifolds of \mathcal{D}). They represent, indeed, gravitational waves having the light as one of possible sources [4–6, 19].

3. Global Solutions

Here, we will give a coordinate-free description of previous local Ricci-flat metrics, so that it becomes clear what variety of different geometries, in fact, is obtained. We will see that with any of the found solutions a pair, consisting of a ζ -complex curve \mathcal{W} and a ζ -harmonic function u on it, is associated. If two solutions are equivalent, then the corresponding pairs, say (\mathcal{W}, u) and (\mathcal{W}', u') , are related by an invertible ζ -holomorphic map $\Phi : (\mathcal{W}, u) \longrightarrow (\mathcal{W}', u')$ such that $\Phi^*(u') = u$. Roughly speaking, the *moduli space* of the obtained geometries is surjectively mapped on the *moduli space* of the pairs (\mathcal{W}, u) .

Further parameters, distinguishing the metrics we are analyzing, are given below. Before that, however, it is worth to underline the following common peculiarities of these metrics

- they have, in the adapted coordinates, a block diagonal form whose upper block does not depend on the last two coordinates so that orthogonal leaves are totally geodesic.
- they possess a non trivial Killing field. Geodesic flows, corresponding to metrics, admitting three dimensional Killing algebras, are non-commutatively integrable. The existence of a non trivial Killing field is obvious from the description of model solution given in next section. For what concerns geodesic flows, they are integrated explicitly for model solution in next section, and the general result follows from the fact that any solution is a pull-back of a model one.

Solutions of the Einstein equations previously described manifest an interesting common feature. Namely, each of them is determined completely by a choice of

- 1) a solution of the wave, or the Laplace equation,

and either by

- 2') a choice of the constant A and one of the branches, for β as function of u , of the tortoise equation

$$\beta + A \ln |\beta - A| = u \quad (6)$$

if $g(Y, Y) \neq 0$, or by

- 2'') a choice of a solution of one of the two equations

$$\left[\mu \left(\partial_y^2 - \partial_x^2 \right) + \mu_y \partial_y - \mu_x \partial_x \right] w = 0, \quad \square \mu = 0 \quad (7)$$

$$\left[\mu \left(\partial_y^2 + \partial_x^2 \right) + \mu_y \partial_y + \mu_x \partial_x \right] w = 0, \quad \Delta \mu = 0 \quad (8)$$

in the case $g(Y, Y) = 0$.

They have a natural *fibred structure* with the Killing leaves as fibers. The wave and Laplace equations, mentioned above in 1), are in fact defined on the two dimensional manifold \mathcal{W} which parameterizes the Killing leaves. These leaves themselves are two dimensional Riemannian manifolds and, as such, are geodesically complete.

For this reason the problem of the extension of described local solutions, is reduced to that of the extension of the base manifold \mathcal{W} . Such an extension should carry a geometrical structure that gives an intrinsic sense to the notion of the wave or the Laplace equation and to equations (7) and (8) on it. A brief description of how this can be done is the following.

3.1. ζ -complex Structures

It is known there exist three different isomorphism classes of two dimensional commutative unitary algebras. They are

$$\mathbb{C} = \mathbb{R}[x] / (x^2 + 1), \quad \mathbb{R}_{(2)} = \mathbb{R}[x] / (x^2), \quad \mathbb{R} \oplus \mathbb{R} = \mathbb{R}[x] / (x^2 - 1)$$

Elements of this algebra can be represented in the form $a + \zeta b$, $a, b \in \mathbb{R}$, with $\zeta^2 = -1, 0$, or 1 , respectively. For a terminological convenience we will call them **ζ -complex numbers**. Of course, ζ -complex numbers for $\zeta^2 = -1$ are just ordinary complex numbers. Furthermore, we will use the unifying notation \mathbb{R}_ζ^2 for the algebra of ζ -complex numbers. For instance $\mathbb{C} = \mathbb{R}_\zeta^2$ for $\zeta^2 = -1$.

In full parallel with ordinary complex numbers, it is possible to develop a ζ -complex analysis by defining **ζ -holomorphic** functions as \mathbb{R}_ζ^2 -valued differentiable functions of the variable $z = x + \zeta y$. Just as in the case of ordinary complex numbers, the function $f(z) = u(x, y) + \zeta v(x, y)$ is ζ -holomorphic *iff* the ζ -Cauchy-Riemann conditions hold

$$u_x = v_y, \quad u_y = \zeta^2 v_x. \quad (9)$$

The compatibility conditions of the above system requires that both u and v satisfy the **ζ -Laplace equation**, that is

$$-\zeta^2 u_{xx} + u_{yy} = 0, \quad -\zeta^2 v_{xx} + v_{yy} = 0.$$

Of course, the ζ -Laplace equation reduces for $\zeta^2 = -1$ to the ordinary Laplace equation, while for $\zeta^2 = 1$ to the wave equation. The operator $-\zeta^2 \partial_x^2 + \partial_y^2$ will be called the **ζ -Laplace operator**.

In the following a **ζ -complex structure** on \mathcal{W} will denote an endomorphism $J : D(\mathcal{W}) \rightarrow D(\mathcal{W})$ of the $C^\infty(\mathcal{W})$ module $D(\mathcal{W})$ of all vector fields on \mathcal{W} , with $J^2 = \zeta^2 I$, $J \neq 0, I$, and vanishing Nijenhuis torsion, i.e., $[J, J]^{FN} = 0$, where

$[\cdot, \cdot]^{FN}$ denotes for the Frölicher-Nijenhuis bracket. A two dimensional manifold \mathcal{W} supplied with a ζ -complex structure is called a ζ - **complex curve**.

Obviously, for $\zeta^2 = -1$ a ζ -complex curve is just an ordinary one dimensional complex manifold (curve).

By using the endomorphism J the ζ -Laplace equation can be written intrinsically as

$$d(J^*du) = 0$$

where $J^* : \Lambda^1(\mathcal{W}) \longrightarrow \Lambda^1(\mathcal{W})$ is the *adjoint to J endomorphism* of the $C^\infty(\mathcal{W})$ module of one forms on \mathcal{W} .

Given a two dimensional smooth manifold \mathcal{W} , an atlas $\{(U_i, \Phi_i)\}$ on \mathcal{W} is called ζ -*complex iff*

- i) $\Phi_i : U_i \longrightarrow \mathcal{W}$, U_i is open in \mathbb{R}_ζ^2
- ii) the transition functions $\Phi_j^{-1} \circ \Phi_i$ are ζ -holomorphic.

Two ζ -complex atlases on \mathcal{W} are said to be *equivalent* if their union is again a ζ -complex atlas.

A class of ζ -complex atlases on \mathcal{W} supplies, obviously, \mathcal{W} with a ζ -complex structure. Conversely, given a ζ -complex structure on \mathcal{W} there exists a ζ -complex atlas on \mathcal{W} inducing this structure. Charts of such an atlas will be called ζ - **complex coordinates** on the corresponding ζ - complex curve. In ζ -complex coordinates the endomorphism J and its adjoint J^* are described by the relations

$$\begin{aligned} J(\partial_x) &= \partial_y, & J(\partial_y) &= \zeta^2 \partial_x \\ J^*(dx) &= \zeta^2 dy, & J^*(dy) &= dx. \end{aligned}$$

If $\zeta^2 \neq 0$, the functions u and v in the equation (9) are said to be **conjugate**.

Alternatively, a ζ -complex curve can be regarded as a two dimensional smooth manifold supplied with a specific atlas whose transition functions

$$(x, y) \longmapsto (\xi(x, y), \eta(x, y))$$

satisfy to ζ -Cauchy-Riemann relations (9).

As it is easy to see, the ζ -Cauchy-Riemann relations (9) imply that

$$\partial_\eta^2 - \zeta^2 \partial_\xi^2 = \frac{1}{\xi_x^2 - \zeta^2 \xi_y^2} (\partial_y^2 - \zeta^2 \partial_x^2)$$

and also

$$\mu (\partial_\eta^2 - \zeta^2 \partial_\xi^2) + \mu_\eta \partial_\eta - \zeta^2 \mu_\xi \partial_\xi = \frac{1}{\xi_x^2 - \zeta^2 \xi_y^2} [\mu (\partial_y^2 - \zeta^2 \partial_x^2) + \mu_y \partial_y - \zeta^2 \mu_x \partial_x].$$

This shows that equation (7) (respectively, (8)) is well-defined on a ζ -complex curve with $\zeta^2 = 1$ (respectively, $\zeta^2 = -1$). The manifestly intrinsic expression for

these equations is

$$d(\mu J^* dw) = 0.$$

We will refer to it as the μ -deformed ζ -Laplace equation.

A solution of the ζ -Laplace equation on \mathcal{W} will be called ζ -**harmonic**. We can see that in the case $\zeta^2 \neq 0$ the notion of *conjugate ζ -harmonic function* is well defined on a ζ -complex curve. In addition, notice that the metric field $d\xi^2 - \zeta^2 d\eta^2$, η being ζ -conjugate with ξ , is canonically associated with a ζ -harmonic function ξ on \mathcal{W} .

A map $\Phi : \mathcal{W}_1 \rightarrow \mathcal{W}_2$ connecting two ζ -complex curves will be called ζ -*holomorphic* if $\varphi \circ \Phi$ is locally ζ -holomorphic for any local ζ -holomorphic function φ on \mathcal{W}_2 . Obviously, if Φ is ζ -holomorphic and u is a ζ -harmonic function on \mathcal{W}_2 , then $\Phi^*(u)$ is ζ -harmonic on \mathcal{W}_1 .

It is worth noting that the *standard ζ -complex curve* is $\mathbb{R}_\zeta^2 = \{(x + \zeta y)\}$, and the *standard ζ -harmonic function* on it is given by x , whose conjugated is y . The pair (\mathbb{R}_ζ^2, x) is **universal** in the sense that for a given ζ -harmonic function u on a ζ -complex curve \mathcal{W} there exists a ζ -holomorphic map $\Phi : \mathcal{W} \rightarrow \mathbb{R}_\zeta^2$ defined uniquely by the relations $\Phi^*(x) = u$ and $\Phi^*(y) = v$, v being conjugated with u .

3.2. Global Properties of Solutions

The above discussion shows that any global solution, that can be obtained by matching together local solutions described in Section 1, is a solution whose base manifold is a ζ -complex curve \mathcal{W} and which corresponds to a ζ -harmonic function u on \mathcal{W} .

A solution of Einstein equations corresponding to $\mathcal{W} \subseteq \mathbb{R}_\zeta^2$, $u \equiv x$ will be called a *model*. Notice that there exist various model solutions due to various options in the choice of parameters appearing in 2') and 2'') at the beginning of this section. An important role played by the model solutions is revealed by the property [18] that

Any solution of the Einstein equation which can be constructed by matching together local solutions described in Section 1 is the pullback of a model solution via a ζ -holomorphic map from a ζ -complex curve to \mathbb{R}_ζ^2 .

We distinguish between the following two qualitatively different cases:

- I metrics admitting a normal three dimensional Killing algebra with two dimensional leaves
- II metrics admitting a normal two dimensional Killing algebra that does not *extend* to a larger algebra having the same leaves and whose distribution orthogonal to the leaves is integrable.

It is worth mentioning that the distribution orthogonal to the Killing leaves is automatically integrable in Case I [17]. In Case II the two dimensionality of the Killing leaves is guaranteed by Proposition 2 of [17].

Any Ricci-flat manifold (M, g) , we are analyzing, is fibered over a ζ -complex curve \mathcal{W}

$$\pi : M \longrightarrow \mathcal{W}$$

whose fibers are the Killing leaves and as such are two dimensional Riemann manifolds of constant Gauss curvature.

Below, we shall call π the *Killing fibering* and assume that its fibers are *connected and geodesically complete*. Therefore, maximal (*i.e., non-extendible*) Ricci-flat manifolds, of the class we are analyzing in the paper, are those corresponding to maximal (*i.e., non-extendible*) pairs (\mathcal{W}, u) , where \mathcal{W} is a ζ -complex curve and u is ζ -harmonic function on \mathcal{W} .

4. Examples

In this Section, we illustrate the previous general results with a few examples using the fact that any solution can be constructed as the pullback of a model solution *via* a ζ -holomorphic map Φ of a ζ -complex curve \mathcal{W} to \mathbb{R}_ζ^2 . Recall that in the pair (\mathcal{W}, u) , describing the so obtained solution, $u = \text{Re}(\Phi)$.

4.1. A Star “Outside” the Universe

The Schwarzschild solution shows a “star” generating a space “around” itself. It is an $\mathfrak{s}(3)$ -invariant solution of the vacuum Einstein equations. On the contrary, its $\mathfrak{s}(2, 1)$ -analogue shows a “star” generating the space only on “one side of itself”. More exactly, the fact that the space in the Schwarzschild universe is formed by a one-parametric family of “concentric” spheres allows one to give a sense to the adverb “around”. In the $\mathfrak{s}(2, 1)$ -case the space is formed by a one-parameter family of “concentric” hyperboloids. The adjective “concentric” means that the curves orthogonal to hyperboloids are geodesics and metrically converge to a singular point. This explains in what sense this singular point generates the space only on “one side of itself”.

4.2. Kruskal-Szekeres Type Solutions

We describe now a family of solutions which are of the *Kruskal-Szekeres type*, namely, that are characterized as being maximal extensions of the local solutions determined by an affine parametrization of null geodesics, and also by the use of more than one interval of monotonicity of $u(\beta)$.

Consider the ζ -complex curve

$$\mathcal{W} = \left\{ (z = x + \zeta y) \in \mathbb{R}_\zeta^2; y^2 - x^2 < 1 \right\}, \quad \zeta^2 = 1$$

and the ζ -holomorphic function $\Phi : \mathcal{W} \rightarrow \mathbb{R}_\zeta^2$

$$\Phi(z) = A \ln \left(|A| z^2 \right) = A \left(\ln \left| A \left(x^2 - y^2 \right) \right| + \zeta \ln \left| \frac{x + y}{x - y} \right| \right).$$

Thus, in the pair (\mathcal{W}, u) the ζ -harmonic function u is given by

$$u = A \ln \left| A \left(x^2 - y^2 \right) \right|.$$

Let us decompose \mathcal{W} in the following way

$$\mathcal{W} = \mathcal{U}_1 \cup \mathcal{U}_2$$

where

$$\begin{aligned} \mathcal{U}_1 &= \left\{ (z = x + \zeta y) \in \mathbb{R}_\zeta^2; 0 \leq y^2 - x^2 < 1 \right\} \\ \mathcal{U}_2 &= \left\{ (z = x + \zeta y) \in \mathbb{R}_\zeta^2; y^2 - x^2 \leq 0 \right\}. \end{aligned}$$

Consider now the solution defined as the pull back with respect to $\Phi|_{\mathcal{U}_1}$ and $\Phi|_{\mathcal{U}_2}$ of the model solutions determined by the following data: in the case of $\Phi|_{\mathcal{U}_1}$, $\mathcal{G} = \mathfrak{s}(3)$ or $\mathcal{G} = \mathfrak{s}(2,1)$, characterized by $F(\vartheta) = \sin^2 \vartheta$ or $F(\vartheta) = \sin h^2 \vartheta$ respectively, $\epsilon_1 = \epsilon_2 = 1$, $A > 0$, and for $\beta(u)$ the interval $]0, A]$. In the case of $\Phi|_{\mathcal{U}_2}$ the same data except for $\beta(u)$ which belongs to the interval $[A, \infty[$. The case $F(\vartheta) = \sin^2 \vartheta$, corresponding to $\mathfrak{s}(3)$, will give the Kruskal-Szekeres solution. The case $F(\vartheta) = \sin h^2 \vartheta$, corresponding to $\mathfrak{s}(2,1)$, will differ from the previous one in the geometry of the Killing leaves, which will now have a negative constant Gaussian curvature. The metric g has the following local form

$$g = 4A^3 \frac{\exp \frac{\beta}{A}}{\beta} \left(dy^2 - dx^2 \right) + \beta^2 \left[d\vartheta^2 + F(\vartheta) d\varphi^2 \right]$$

with singularity $\beta = 0$ occurring at $y^2 - x^2 = 1$.

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References

- [1] Aliev B. and Leznov A., *Exact Solutions of the Vacuum Einstein's Equations Allowing for two Noncommutative Killing Vectors Type G_2II of Petrov classification*, J. Math. Phys. **33** (1992) 2567-2573.
- [2] Bächtold M., *Ricci Flat Metrics with Bidimensional Null Orbits and Non-Integrable Orthogonal Distribution*, Diff. Geom. Appl. **25** (2007) 167-176
- [3] Belinsky V. and Zakharov V., *Integration of the Einstein Equations by Means of the Inverse Scattering Problem Technique and Construction of the Exact Soliton Solutions*, Sov. Phys. JETP **48** (1978) 985-994.
- [4] Canfora F., Vilasi G. and Vitale P., *Nonlinear Gravitational Waves and Their Polarization*, Phys. Lett. B **545** (2002) 373-378.
- [5] Canfora F., Vilasi G. and Vitale P., *Spin-1 Gravitational Waves*, Int. J. Mod. Phys. B **18** (2004) 527-540.
- [6] Canfora F. and Vilasi G., *Spin-1 Gravitational Waves and Their Natural Sources*, Phys. Lett. B **585** (2004) 193-199.
- [7] Catalano-Ferraioli D. and Vinogradov A., *Ricci-flat 4-Metrics with Bidimensional Null Orbits, Part I. General Aspects and Nonabelian Case*, Acta. Appl. Math. **92** (2006) 209-225; *Ricci-flat 4-Metrics with Bidimensional Null Orbits, Part II. Abelian Case*, Acta. Appl. Math. **92** (2006) 226-239.
- [8] Chinea F., *New First Integral for Twisting Type-N Vacuum Gravitational Fields with Two Non-Commuting Killing Vectors*, Class. Quantum Grav. **15** (1998) 367-371.
- [9] Einstein A. and Rosen N., *On Gravitational Waves*, J. Franklin Inst. **223** (1937)43-54.
- [10] Hallisoy M., *Studies in Space-Times Admitting Two Spacelike Killing Vectors*, J. Math. Phys. **29** (1988) 320-326.
- [11] Law P., *Neutral Einstein Metrics in Four Dimensions*, J. Math. Phys. **32** (1991) 3039-3042.
- [12] Matsushita Y., *On Euler Characteristics of Compact Einstein 4-Manifolds of Metric Signature $(+ + - -)$* , J. Math. Phys. **22** (1981) 979-982.
- [13] Matsushita Y., *Thorpe-Hitchin Inequality for Compact Einstein 4-Manifolds of Metric Signature $(+ + - -)$ and the Generalized Hirzebruch Index Formula*, J. Math. Phys. **24** (1983) 36-40.
- [14] Peres A., *Some Gravitational Waves*, Phys. Rev. Lett. **3** (1959) 571-572.
- [15] Sparano G. and Vilasi G., *Noncommutative Integrability and Recursion Operators*, J. Geom. Phys. **36** (2000) 270-284.
- [16] Sparano G., Vilasi G. and Vinogradov A., *Gravitational Fields with a Non-Abelian, Bidimensional Lie Algebra of Symmetries*, Phys. Lett. B **513** (2001) 142-146.
- [17] Sparano G., Vilasi G. and Vinogradov A., *Vacuum Einstein Metrics with Bidimensional Killing Leaves. I. Local Aspects*, Diff. Geom. Appl. **16** (2002) 95-120.
- [18] Sparano G., Vilasi G. and Vinogradov A., *Vacuum Einstein Metrics with Bidimensional Killing Leaves. II. Global Aspects*, Diff. Geom. Appl. **17** (2002) 15-35.
- [19] Vilasi S., Canonico R., Sparano G. and Vilasi G., *Is the Light too Light?*, J. Geom. Symm. Phys. **19** (2010) 87-98.