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QUANTUM INTEGRABILITY AND COMPLETE SEPARATION OF VARIABLES FOR PROJECTIVELY EQUIVALENT METRICS ON THE TORUS

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Abstract. Let two Riemannian metrics g and $\bar{g}$ on the torus T^n have the same geodesics (considered as unparameterized curves). Then we can construct invariantly n commuting differential operators of second order. The Laplacian Δ_g of the metric g is one of these operators. For any $x \in T^n$, consider the linear transformation G of T_xT^n given by the tensor $g^{i\alpha}\bar{g}_{\alpha j}$. If all eigenvalues of G are different at one point of the torus then they are different at every point; the operators are linearly independent and we can globally separate the variables in the equation $\Delta_g f = \mu f$ on this torus.

1. Commuting Operators for Projectively Equivalent Metrics

Let g and $\bar{g}$ are two C^2-smooth Riemannian metrics on some manifold M^n. They are **projectively equivalent** if they have the same geodesics considered as unparameterized curves.

The problem of describing projectively equivalent metrics was stated by Beltrami in [1]. Locally, in the neighborhood of so-called sable points, it was essentially solved by Dini [3] for surfaces and by Levi-Civita [4] for manifolds of arbitrary dimension. Denote by G the tensor $g^{i\alpha}\bar{g}_{\alpha j}$. In invariant terms, G is the fiberwise-linear mapping $G\colon TM^n \to TM^n$ such that its restriction to any tangent space $T_{x_0}M^n$ is the linear transformation of $T_{x_0}M^n$ satisfying the following condition: for any vectors $\xi, \nu \in T_{x_0}M^n$, the scalar product $g(G(\xi), \nu)$ of the vectors $G(\xi)$ and ν in g is equal to the scalar product $\bar{g}(\xi, \nu)$ of the vectors ξ and ν in $\bar{g}$.