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NECESSARY CONDITIONS FOR A SUPERDIFFERENTIABLE SUPERCURVE TO BE A WEAK MINIMUM RELATIVE TO TWO SUB-SUPERMANIFOLDS

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Abstract. Let L defines a regular problem in the calculus of variations on supermanifolds. The necessary conditions for a piecewise superdifferentiable supercurve C in sense of Rogers to be a weak local minimum relative to two sub-supermanifolds are given.

Let V be a supervector space [3], V^* be the dual supervector space [5], M be a supermanifold in the sense of Rogers [7] and $T(M)$ be the tangent superspace or superbundle [5] over M.

Let us consider only algebras over the real numbers. For each positive integer L, B_L [7] will denote the Grassmannian algebra over the real numbers with generators $1^L, \beta_1^L, \dots, \beta_L^L$ and relations

$$1^L \cdot \beta_i^L = \beta_i^L \cdot 1^L = \beta_i^L \quad i = 1, \dots, L\,,$$
$$\beta_i^L \cdot \beta_j^L = -\beta_j^L \cdot \beta_i^L \quad i, j = 1, \dots, L\,.$$

B_L is a graded algebra [8] and can be written as a direct sum [7]

$$B_L = (B_L)_0 \oplus (B_L)_1$$

where $(B_L)_0$ and $(B_L)_1$ are the even and the odd parts of (B_L) respectively. We consider the (m, n)-dimensional supereuclidean space $B_L^{m,n} = (B_L)_0^m \oplus (B_L)_1^n$ [7] with $L > n$. Let M_L denote (following Kostant [6]) the set of finite sequences of positive integers $\mu = (\mu_1, \dots, \mu_k)$ with $1 \leq \mu_1 < \cdots < \mu_k \leq L$. M_L includes also the sequence with no elements, which is denoted by ϕ. As it follows from [6] for each μ in M_L

$$\beta_\mu^{(L)} = \beta_{\mu_1}^{(L)} \cdots \beta_{\mu_k}^{(L)}, \quad k = 1, \dots, L$$

and

$$\beta_{\phi}^{(L)} = 1^{(L)}$$

a typical element b of B_L may be expressed as

$$b = \sum_{\mu \in M_L} b^{\mu} \beta_{\mu}^{(L)}$$

where the coefficients b^{μ} are real numbers. With the norm on B_L defined by

$$\|b\| := \sum_{\mu \in M_L} |b^{\mu}|$$

B_L is a Banach algebra [7].

We consider the body map (in de Witt's terminology [5])

$$\varepsilon_L : B_L \to \mathbb{R}$$

given by

$$\varepsilon_L(b) = b^{\phi}.$$

As explained in [3] $B_L^{m+n} = B_L^{m,n} \oplus B_L^{n,m}$ where $n = 2r$, one can define the following scalar product

$$\langle v, w \rangle = x^1 y^1 + \cdots + x^m y^m + \theta^1 {\theta'}^{r+1} + \cdots + \theta^r {\theta'}^n \\ - \theta^{r+1} {\theta'}^1 - \cdots - \theta^n {\theta'}^r$$

for all $v = (x^1, \ldots, x^m, \theta^1, \ldots, \theta^n)$,

and $w = (y^1, \ldots, y^m, {\theta'}^1, \ldots, {\theta'}^n) \in B_L^{m+n}$.

Definition 1. (Rogers [7]) *A function $f : B_L^{m,n} \to B_L$ is called a superdifferentiable function if there exist $f_\mu \in C^\infty(\mathbb{R}^m, \mathbb{R})$ such that:*

$$f(x, \theta) = \sum_{\mu \in M_n} f_\mu(x) \theta^\mu$$

where $M_n = \{(\mu_1, \ldots, \mu_n);\ 1 \le \mu_1 < \cdots < \mu_n \le n\}$ [6].

Let M be a Hausdorff topological space. Then: (a) an (m, n) chart on M over B_L is a pair (U, ψ) with U an open set of M and ψ a homeomorphism of U onto an open subset of $B_L^{m,n}$ and (b) an (m, n) superdifferentiable structure on M over B_L is a collection $\{(U_\alpha, \psi_\alpha);\ \alpha \in \Lambda\}$ of (m, n) charts on M such that (i) $M = \cup_{\alpha \in \Lambda} U_\alpha$; (ii) for each pair α, β in Λ the mapping $\psi_\beta \circ {\psi_\alpha}^{-1}$ is a superdifferentiable function of $\psi_\alpha(U_\alpha \cap U_\beta)$ onto $\psi_\beta(U_\alpha \cap U_\beta)$, and (iii) the collection $\{(U_\alpha, \psi_\alpha);\ \alpha \in \Lambda\}$ is a maximal collection of open charts for which (i) and (ii) hold.

Definition 2. *An (m,n)-dimensional superdifferentiable supermanifold over B_L is a Hausdorff topological space M with an (m,n) superdifferentiable structure over B_L.*

Definition 3. (de Witt [5]) *A subset M' of a supermanifold M of dimension (m,n) is called a sub-supermanifold of dimension (m',n'), $m \geq m'$, $n \geq n'$, if M' is contained in the union of a set $\{(U,\psi)\}$ of charts each of which has the property that, for all $(x,\theta) \in U \cap M'$, $\psi(x,\theta) = (x^1,\dots,x^{m'},a^{m'+1},\dots,a^m,\theta^1,\dots,\theta^{n'},\eta^{n'+1},\dots,\eta^n)$ where $(a^{m'+1},\dots,a^m,\eta^{n'+1},\dots,\eta^n)$ is a fixed element of $B_L^{m-m',n-n'}$ depending on the chart in question.*

The pairs $\{(U',\psi')\}$ where $U' = U \cap M'$ and $\psi'(x,\theta) = (x^1,\dots,x^{m'}, \theta^1,\dots,\theta^{n'})$ constitute an atlas for M'.

Example 1. *Let us consider the (m,n)-dimensional supereuclidean space $B_L^{m,n}$. It is an (m,n)-dimensional superdifferentiable supermanifold over B_L from the Definition 2. We consider the subset $S_L^{m-1,n-2}$ of $B_L^{m,n}$, where $S_L^{m-1,n-2} = \{(x,\theta) \in B_L^{m,n};\ (x^1)^2 + \cdots + (x^m)^2 + 2\theta^1\theta^{r+1} + \cdots + 2\theta^r\theta^n = 1 + 2\beta_1\beta_{r+1} + \cdots + \beta_r\beta_n\}$ and we conclude that $S_L^{m-1,n-2}$ is an $(m-1,n-2)$-dimensional sub-supermanifold of $B_L^{m,n}$.*

Definition 4. *The function $C\colon [a,b] \to M$ is called a superdifferentiable supercurve [3] if the functions $x^i \circ C$ for all $i \in [1,m]$ and $\theta^\alpha \circ C$ for all $\alpha \in [1,n]$ are superdifferentiable [7], the functions $\varepsilon_L \circ x^i \circ C$ for all $i \in [1,m]$ and $\varepsilon_L \circ \theta^\alpha \circ C$ for all $\alpha \in [1,n]$ are differentiable in $\mathbb{R}$ and (x^i,θ^α) are the coordinates of a point $p \in M$.*

Definition 5. *Let L be a superdifferentiable function on $T(M) \times B_L$ and we make distinction between this superdifferentiable function L and the positive integer L. Then L defines a superdifferentiable map $L'\colon T(M) \times B_L \to T^*(M) \times B_L$ called the Legendre supertransformation, which is given in local coordinates by $x^i \circ L' = x^i$ for all $i \in [1,m]$, $\theta^\alpha \circ L' = \theta^\alpha$ for all $\alpha \in [1,n]$, $y^i \circ L' = \dfrac{\partial L}{\partial x^i}$ for all $i \in [1,m]$, $\delta^\alpha \circ L' = \dfrac{\partial L}{\partial \theta^\alpha}$ for all $\alpha \in [1,n]$ and $t \circ L' = t$.*

Definition 6. *If the Legendre supertransformation is an immersion [5] of $T(M) \times B_L$ into $T^*(M) \times B_L$, then the function L will be called a regular super-Lagrangian.*

Definition 7. *If the Legendre supertransformation is an immersion, the map ${L'}^{-1}$ comes locally in a similar way from a function H on $T^*(M) \times B_L$ is called super-Hamiltonian:*

$$H(y,\delta) = \langle {L'}^{-1}(y,\delta),(y,\delta)\rangle - L \circ \mathcal{L}^{-1}(y,\delta)\,.$$

The function $E = H \circ L'$ is globally well-defined on $T(M) \times B_L$.

Theorem 1. *If M is a Riemannian supermanifold and $L(v,t) = \frac{1}{2}\langle v, v\rangle$, then L is a regular super-Lagrangian and L' coincides on each tangent superspace [5] with the map of $T_q(M) \to T_q^*(M)$ given by the scalar product introduced in [3]. Furthermore, in this case $E = H \circ L' = L$.*

Proof: This is proved in [4]. □

Definition 8. *Let $C\colon [a,b] \to M$ be a superdifferentiable supercurve on M. Then C determines a supercurve $\tilde{C}$, on $T(M) \times B_L$ defined by*

$$\tilde{C}(t) = (C'(t), t)$$

for each $t \in [a,b]$. Therefore, we can consider the integral

$$I(C) = \int_a^b L\left(\tilde{C}(t)\right) \mathrm{d}t\,.$$

Let C_j and C_j^1 be the restrictions of C and C^1 respectively to the interval $[s_j, s_{j+1}]$, where $a = s_0 < \cdots < s_r = b$ and $W \subset M$, C_j and C_j^1 be superdifferentiable supercurves of $(s_j + \varepsilon, s_{j+1} - \varepsilon)$ into W.

Definition 9. *A supercurve C is called weak local minimum if there are W and $\varepsilon > 0$ such that $\varepsilon_L(I(C)) \leq \varepsilon_L(I(C^1))$ for all piecewise superdifferentiable supercurves satisfying*

$$C^1(a) = C(a) \quad \text{and} \quad C^1(b) = C(b). \tag{1}$$

Proposition 1. *Let C be a weak local minimum of L. Then at every point t where C is superdifferentiable the tangent supervector $Y_q = C'(t)$ satisfies*

$$Y_q \lrcorner \mathrm{d}\omega_q = 0 \tag{2}$$

for $\theta^\alpha(t) = t\left(\bar{\delta}^\alpha(t) + \bar{\delta}^{\alpha+r}(t)\right)$ and $\theta^{\alpha+r}(t) = t\left(\bar{\delta}^{\alpha+r}(t) - \bar{\delta}^\alpha(t)\right)$ for all $\alpha \in \{1, \ldots, r\}$ and $(y, \bar{\delta})$ are coordinates on $(B_L^{m+n})^*$ where

$$e_i \lrcorner e^{j_1} \wedge \cdots \wedge e^{j_r} = \begin{cases} 0 & \text{if } i \neq j_k \text{ for any } k \\ (-1)^{(i)+k-1} e^{j_1} \wedge \cdots \wedge e^{j_{k-1}} \wedge e^{j_{k+1}} \wedge \cdots \wedge e^{j_r} & \text{if } i = j_k \end{cases}$$

and i is 0 if $e_i \in B_L^{m,n}$ or 1 if $e_i \in B_L^{n,m}$ and where $(e_i)_{i=1,\ldots,m+n}$ is a basis of B_L^{m+n} and $(e^j)_{j=1,\ldots,m+n}$ is a basis of $(B_L^{m+n})^*$.

Proof: This is proved in [4]. □

Theorem 2. ([4]) *Let L define a regular problem in the calculus of variations on supermanifolds. A necessary condition that a piecewise superdifferentiable supercurve C in sense of Rogers to be a weak local minimum for L is that C is superdifferentiable and $\tilde{C}$ is an integral supercurve of X where X is defined by*

$$X \lrcorner \, \mathrm{d}\sigma = 0, \qquad \omega = \mathrm{d}L, \qquad \langle X, \mathrm{d}t \rangle = 1 \tag{3}$$

along with the superform $\sigma = L'^{*}\omega$ to be well defined on $T(M) \times B_L$ and L' to be an immersion of $T(M) \times B_L$ into $T^*(M) \times B_L$.

Proof: This is proved in [4]. □

Definition 10. *Let N_1 and N_2 be two sub-supermanifolds of M. A superdifferentiable supercurve is called a weak minimum of L relative to N_1 and N_2 if it satisfies the conditions of Definition 9 with*

$$C^1(a) \in N_1 \quad \textit{and} \quad C^1(b) \in N_2. \tag{4}$$

Theorem 3. *Let C be a weak mimimun relative to N_1 and N_2. Then C is an extremal of L and furthermore*

$$\langle v_1, \bar{C}(a) \rangle = \langle v_2, \bar{C}(b) \rangle = 0 \tag{5}$$

for all $v_1 \in T_{C(a)}N_1$ and for all $v_2 \in T_{C(b)}N_2$.

Proof: The first part of the Theorem 3 is easy: if C is a weak minimun relative to N_1 and N_2, it is certainly a weak minimum in the sense of Definition 9. Thus Theorem 2 implies that C is an extremal and it suffices to prove one of the equalities in (3). Let $h = (x^1, \dots, x^m, \theta^1, \dots, \theta^n)$ be a coordinate system in a neighborhood U of $C(b)$ such that $N_2 \cap U$ is the set of points, where $x^{k+1} = \cdots = 0$, $\theta^{l+1} = \cdots = 0$ with $l \leq r$ or $l \geq r$. Let the supervector v_2 be given as $v_2 = a_1 \left(\frac{\partial}{\partial x^1}\right)_{C(b)} + \cdots + a_k \left(\frac{\partial}{\partial x^k}\right)_{C(b)} + \eta_1 \left(\frac{\partial}{\partial \theta^1}\right)_{C(b)} + \cdots + \eta_l \left(\frac{\partial}{\partial \theta^l}\right)_{C(b)}$. Choose t_0 close enough to b so that $C(t) \in U$ for $t_0 \leq t \leq b$. Let $\bar{X}^1(t), \dots, \bar{X}^k(t)$, $\bar{X}'^1(t), \dots, \bar{X}'^l(t)$ be superdifferentiable functions with $\bar{X}^i(t) = 0$ for $i = 1, \dots, k$ and for $t \leq t_0$ and $\bar{X}'^i(t) = 0$ for $i = 1, \dots, l$, $t \leq t_0$ and $\bar{X}^i(b) = a_i$ $(i = 1, \dots, k)$ and $\bar{X}^i(b) = 0$ for $i > k$ and $\bar{X}'^i(b) = \eta_i$ $(i = 1, \dots, l)$ and $\bar{X}'^i(b) = 0$ for $i > l$. Choose ε' so small that $x^1(t) + s\bar{X}_1(t), \dots, x^m(t) + s\bar{X}_m(t), \theta^1(t) + s\bar{X}'_1(t), \dots, \theta^n(t) + s\bar{X}'_n(t) \in h(U)$ for $t_0 \leq t \leq b$ and $|s| \leq \varepsilon'$ where $h \circ C(t) = x^1(t), \dots, x^m(t), \theta^1(t), \dots, \theta^n(t)$. Let K be the map of the rectangle $D = \{|s| \leq \varepsilon';\, a \leq t \leq b\}$ into M defined by $K(s,t) = C(t)$ for $a \leq t \leq t_0$ and $x^i \circ K(s,t) = x^i(t) + s\bar{X}^i(t)$ and $\theta^\alpha \circ K(s,t) = \theta^\alpha(t) + s\bar{X}'^\alpha(t)$ for $t_0 \leq t \leq b$. Then as before, we have $\varepsilon_L\left(\int_{[a,b]}\bar{C}^*(\omega)\right) \geq \varepsilon_L\left(\int \bar{K}^*(s,\cdot)\omega\right)$ for sufficiently small $|s|$. As in the proof of Lemma 2.2 from [9] and Proposition 1 this implies that, for $s > 0$ $\varepsilon_L\left(\int_{[0,s]}\bar{K}^*(\cdot,t)\omega\right) \geq \varepsilon_L\left(\int_{D_+}\bar{K}^*\,\mathrm{d}\omega\right)$. Dividing by s and letting $s \to 0$ we obtain, since C is an extremal $\varepsilon_L\Big(\Big\langle \bar{K}_*(\cdot,t)\left(\frac{\partial}{\partial s}\right)_{(0,b)} \Big|\ \omega_{\bar{C}(b)}\Big\rangle\Big) \geq 0$ where $D_+ = \{0 \leq s \leq \varepsilon';\ a \leq t \leq b\}$.

Doing the same for the negative s we obtain

$$\varepsilon_L\Big(\Big\langle \bar{K}_*(\cdot,t)\left(\frac{\partial}{\partial s}\right)_{(0,b)} \Big|\ \omega_{\bar{C}(b)}\Big\rangle\Big) \leq 0\,.$$

Thus $\varepsilon_L\left(\langle \bar{K}_*(\cdot,t)\left(\frac{\partial}{\partial s}\right)_{(0,b)} \mid \omega_{\bar{C}(b)}\rangle\right) = 0$. But

$$\varepsilon_L\left(\bar{K}_*(\cdot,t)\left(\frac{\partial}{\partial s}\right)_{(0,b)}\right) = a_1\left(\frac{\partial}{\partial x^1}\right)_{C(b)} + \cdots + a_k\left(\frac{\partial}{\partial x^k}\right)_{C(b)}$$

and

$$\omega_{\bar{C}(t)} = \sum_{i=1}^{m} y^i(b)\,\mathrm{d}x_b^i - \sum_{\alpha=1}^{r}\left(\delta^\alpha(b)\,\mathrm{d}\theta_b^{\alpha+r} - \delta^{\alpha+r}(b)\,\mathrm{d}\theta_b^\alpha\right) - H\,\mathrm{d}t$$

where $\bar{C}(b) = \sum_{i=1}^m y^i(b)\,\mathrm{d}x_b^i - \sum_{\alpha=1}^r\left(\delta^\alpha(b)\,\mathrm{d}\theta_b^{\alpha+r} - \delta^{\alpha+r}(b)\,\mathrm{d}\theta_b^\alpha\right), b)$. Thus

$$\begin{aligned}\varepsilon_L\Big(\Big\langle \bar{K}_*(\cdot,t)\left(\frac{\partial}{\partial s}\right)_{(0,b)} \Big|\ \omega_{\bar{C}(b)}\Big\rangle\Big) &= \sum_{i=1}^{k} y^i(b)a_i + \sum_{\alpha=1}^{l} -\delta^{\alpha+r}(b)\eta^\alpha\\ &= \langle v_2, \bar{C}(b)\rangle = 0\end{aligned}$$

if $l \leq r$ and

$$\begin{aligned}&\varepsilon_L\Big(\Big\langle \bar{K}_*(\cdot,t)\left(\frac{\partial}{\partial s}\right)_{(0,b)} \Big|\ \omega_{\bar{C}(b)}\Big\rangle\Big)\\ &\qquad= \sum_{i=1}^{k} y^i(b)a_i + \sum_{\alpha=1}^{l-r}\delta^\alpha(b)\eta^{\alpha+r} - \sum_{\alpha=1}^{r}\delta^{\alpha+r}(b)\eta^\alpha\\ &\qquad= \langle v_2, \bar{C}(b)\rangle = 0\end{aligned}$$

if $l \geq r$, which proves Theorem 3. □

Remark 1. *If L the kinetic superenergy associated to the Riemannian supermetric g [1], [5], then L is given by [4]*

$$L = \frac{1}{2}\sum_{i,j=1}^{m} g_{ij}\dot{x}^i\dot{x}^j + \frac{1}{2}\sum_{\alpha,\beta=1}^{n}\bar{g}_{\alpha\beta}\dot{\theta}^\alpha\dot{\theta}^\beta$$

while condition (3) says that $C'(a)$ *is orthogonal to* $T_{C(a)}(N_1)$ *relative to inner product on* $T_{C(a)}(M)$ *given by* g *and* $C'(b)$ *is orthogonal to* $T_{C(b)}(N_2)$*, i. e.* $\langle v, C'(a)\rangle = 0$ *for* $v \in T_{C(a)}(N_1)$ *and* $\langle v, C'(b)\rangle = 0$ *for* $v \in T_{C(b)}(N_2)$.

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