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VARIATIONAL SYMMETRIES AND LIE REDUCTION FOR FROBENIUS SYSTEMS OF EVEN RANK

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Abstract. Let $\mathcal{I}$ be Frobenius system of even rank. Consider a closed two-form $\Pi \in \mathcal{I} \wedge \mathcal{I}$ of maximal rank. A vector field X such that $\mathcal{L}_X \Pi = 0$ is called a symmetry of Π. We determine the relationship between the solvable Lie group of symmetries of Π and the rank of the reduced system obtained from $\mathcal{I}$ by Lie reduction. For an Euler–Lagrange system of ODE's with the corresponding Lagrangian L, Π can be taken to be the differential of the Poincaré–Cartan form η_L. A symmetry of $\Pi = \mathrm{d}\eta_L$ is a variational symmetry of the Lagrangian L. A proof of Noether's theorem for Frobenius systems of even rank is provided.

1. Introduction

This paper, like many others mentioned below, treats Lie symmetry method for systems of ordinary differential equations (ODE's) in the framework of exterior differential systems. Systems of differential equations in one independent variable may be interpreted as Pfaffian systems of codimension one. Such systems are examples of Frobenius (completely integrable) systems.

Lie's symmetry method, named after the famous Norwegian mathematician Sophus Lie, is one of the most successful methods for finding explicit solutions of systems of differential equations [10]. It unifies numerous methods used to integrate special types of equations, such as separable equations, homogeneous equations, Euler's equations, linear equations, Bernoulli equations, and many others. Lie's method has even more important applications into partial differential equations, but we will not discuss this aspect here. It was Sophus Lie who first realized that if one can associate a solvable s-parameter group of symmetries to an n^{th}-order ordinary differential equation, then the equation can be reduced to an $(n-s)^{\text{th}}$-order differential equation in the sense that one can recover by quadratures the general solution of the original equation from the general solution of the reduced equation; if $n = s$, then the general solution can be explicitly written out in terms of quadratures. Lie was concerned with point symmetries since his focus was on the practical implementation of his method which might have been the reason why he did not study more general types of transformations — it turns out that the task of finding a solvable Lie group of intrinsic contact symmetries for a system of ODE's is equivalent to solving the system! In many instances one can eyeball some point symmetries and others can be obtained solving an over-determined system of equations. On the other hand, one of the advantages of considering more general types symmetries in more general setting is that this assumption makes the theory easier (surprisingly). It was Cartan who first proposed to study Lie's methods in the framework of exterior differential systems. He, like Lie, never bothered himself to to write down the general theorems underlying Lee's method. This gap was filled by a number of papers in 70's, 80's, and 90's of the past century. Lie symmetry method have laid long time dormant since the time of Cartan until Ovsiannikov [12] begun his extensive investigations of this classical subject. There is a large number of papers mostly by Soviet mathematicians that deal with various aspects of Lie's method and include computation of symmetry groups of point, contact, and generalized symmetries for many particular classes of physically significant systems of differential equations. The reader can find a summary of these results in [9]. One should also not forget to mention the outstanding book of Bluman and Cole [3]. Modern treatment of Lie's method can be found in [11], where the reader can also find a large list of classical and more recent references. Various extension of Lie's method to differential systems in the spirit of Cartan's ideas are found for example in [7, 5, 4, 2], and [8] (this list is by no means exhaustive).

Symmetries of Lagrangians are called variational symmetries. With the use of a one-parameter variational symmetry group one can reduce the order of the associated Euler–Lagrange equation by two (see, for instance, Olver [11], Theorem 4.17). The easiest way to explain this phenomena is to note that a variatonal symmetry is also a symmetry of the Euler–Lagrange system and recall that, thanks to Emily Noether, we know there is a one-to-one correspondence between variational symmetries and conservation laws. Thus, one can use both the symmetry and the conservation law to reduce the order of the Euler–Lagrange equation. Now consider s-parameter solvable variational symmetry group of a Lagrangian L. Variational symmetries are also symmetries of the Euler–Lagrange equation and so using the Lie reduction method we can reduce the Euler–Lagrange equation by s orders. Due to Noether's theorem, we have, in addition, s conservation laws available to us for further reduction. But, some of them may have been already used in the reduction process using the symmetries and so, in general, the reduced equation will have order from s to $2s$ lower then the original Euler-lagrange equation. The precise order of reduction is determined by Theorem 4.5 which is the main result of this paper. In fact, we establish a more general result: Let $\mathcal{I}$ be Frobenius system of even rank. Consider a closed two-form $\Pi \in \mathcal{I} \wedge \mathcal{I}$ of maximal rank. A vector field X such that $\mathcal{L}_X \Pi = 0$ is called a symmetry of Π. We determine the exact relationship between a solvable Lie group of symmetries of Π and the rank of the reduced system obtained from $\mathcal{I}$ by Lie reduction. In the course of proving Theorem 4.5 we will provide a simple proof of Noether's theorem for Frobenius systems based on Cartan's formula

$$\mathcal{L}_X \omega = X \lrcorner \, \mathrm{d}\omega + \mathrm{d}(X \lrcorner \, \omega).$$

The usefulness of this formula for this paper cannot be overestimated. We will also explain the process of reduction based only on linear algebra operations combined with the use of a slightly modified de Rham homotopy operator. Another interesting question is under what conditions does the reduced equation admit a variational principle? This question was solved in [1].

2. Preliminaries and Definitions

A **Pfaffian system** $\mathcal{I}$ of rank r on an m-dimensional manifold M is a module over the ring $C^\infty(M)$ of smooth functions on M generated by r one-forms ω_1, ω_2, …, ω_r linearly independent at each point of the manifold. We write $\mathcal{I} = \{\omega_1, \ldots, \omega_r\}$. A vector field X is called an **(infinitesimal) symmetry** of $\mathcal{I}$ if the Lie derivative $\mathcal{L}_X \mathcal{I} \subseteq \mathcal{I}$. The set of all symmetries of $\mathcal{I}$ forms a Lie algebra called **symmetry algebra** of $\mathcal{I}$. We call the infinitesimal symmetry X *trivial* or a **Cauchy characteristic vector field** if $X \lrcorner \, \mathcal{I} = \{0\}$, otherwise we say that X is *non-trivial*. More generally, a vector field X is a Cauchy characteristic of a **differential ideal** $\mathcal{I}$ if $X \lrcorner \, \mathcal{I} \subseteq \mathcal{I}$. Two infinitesimal symmetries of $\mathcal{I}$ are called **equivalent** if their difference is a trivial symmetry. A smooth function f on M is called a **conservation law** if $\mathrm{d}f \in \mathcal{I}$. A constant function is called a *trivial conservation law* . Two conservation laws are said to be *equivalent* if they differ by a constant. We say that two Pfaffian systems $(M, \mathcal{I})$ and $(\bar{M}, \bar{\mathcal{I}})$ are *equivalent* if there is a diffeomorphism $\psi\colon M \to \bar{M}$ such that $\psi^*(\bar{\mathcal{I}}) \subseteq \mathcal{I}$.

A **Frobenius system $\mathcal{I}$ of rank** r on an m-dimensional manifold M is a Pfaffian system $\mathcal{I} = \{\omega_1, \ldots, \omega_r\}$ of rank r where $\omega_1, \ldots, \omega_r$ are pointwise linearly independent one-forms satisfying

$$\mathrm{d}\omega_i = \sum_{j=1}^{r} \alpha_i^j \wedge \omega_j, \quad \text{for all} \quad 1 \le i \le r$$

for some one-forms α_i^j. Frobenius theorem implies that two Frobenius systems are locally equivalent if and only if they have the same rank and the two manifolds M and $\bar{M}$ have the same dimension. An **integral manifold** of $\mathcal{I}$ is an immersed submanifold of $\psi\colon N \to M$ such that $\psi^*(\mathcal{I}) = 0$. For a Frobenius system $\mathcal{I}$ of rank r the maximum dimension of an integral manifold is $m - r$. For the given point $p \in M$ there is a local coordinate system $(x^1, \ldots, x^m)$ such that $\mathcal{I}$ is locally generated by the one-forms $\mathrm{d}x^1, \ldots, \mathrm{d}x^r$, and so locally the integral manifolds of $\mathcal{I}$ of dimension $m - r$ are given by $x^1 = a_1$, $x^2 = a_2, \ldots, x^r = a_r$ where $a_1, a_2, \ldots, a_r$ are real constants. By *integrating a Frobenius system* we mean finding all local integral manifolds. It is clear that this process is equivalent with finding r functionally independent local conservation laws for $\mathcal{I}$.

Definition 1. *We say that a system of independent one-forms* $\omega_1, \ldots, \omega_r$, $\omega_{r+1}, \ldots \omega_{r+s}$ *is reducible to* $\omega_1, \ldots \omega_r$ *if*

$$\mathrm{d}\omega_i \equiv 0 \mod \{\omega_1, \ldots, \omega_r, \ldots, \omega_{i-1}\} \quad \textit{for all} \quad r+1 \le i \le r+s\,. \tag{1}$$

This definition generalizes the classical notion of reducibility of one system of ODE's to another system ODE's, as well as the notion of a *solvable structure* introduced in [8]. In fact, if $\mathcal{I}$ is reducible to $\{0\}$, then $\mathcal{I}$ is a solvable structure as defined in [8].

To explain the notion of reducibility, suppose that the reduced system $\mathcal{I}' = \{\omega_1, \ldots \omega_r\}$ is Frobenius and assume that we can explicitly write out the general solution of $\mathcal{I}'$. In other words assume that we know r independent conservation laws f^i and so $\mathcal{I}' = \{\mathrm{d}f^1, \ldots, \mathrm{d}f^r\}$. Let $(f^1, \ldots, f^r, x^1, \ldots, x^p)$ be a local coordinate system on M. Consider the first equation in (1)

$$\mathrm{d}\omega_{r+1} = 0 \mod \{\mathrm{d}f^1, \ldots, \mathrm{d}f^r\}\,. \tag{2}$$

We write

$$\omega_{r+1} = \sum_{i=1}^{p} G_i(f^k, x^l)\,\mathrm{d}x^i + \sum_{j=1}^{r} F_j(f^k, x^l)\,\mathrm{d}f^j \tag{3}$$

and so

$$\mathrm{d}\omega_{r+1} \equiv \sum_{i=1}^{p}\sum_{j=1}^{p} \frac{\partial G_i}{\partial x^j}(f^k, x^l)\,\mathrm{d}x^j \wedge \mathrm{d}x^i \mod \{\mathrm{d}f^1, \ldots, \mathrm{d}f^r\}\,. \tag{4}$$

From (2) and (4) we conclude

$$\frac{\partial G_i}{\partial x^j} - \frac{\partial G_j}{\partial x^i} = 0 \qquad \text{for all} \quad 1 \le i, j \le p\,. \tag{5}$$

We now define a functor H that maps a one-form ω to the function

$$H(\omega) = \int_0^1 (V \lrcorner\, \omega)(f^1, \dots, f^r, \lambda x^1, \dots, \lambda x^p)\,\frac{\mathrm{d}\lambda}{\lambda} \qquad \text{where} \quad V = \sum_{i=1}^{p} x^i \frac{\partial}{\partial x^i}\,.$$

Notice that $H(\omega)$ is just a slight modification of the de Rham homotopy operator. In particular

$$H(\omega_{r+1}) = \int_0^1 \sum_{j=1}^{p} x^j G_j(f^1, \dots, f^r, \lambda x^1, \dots, \lambda x^p)\,\mathrm{d}\lambda$$

and we set $f^{r+1} = H(\omega_{r+1})$. Using (5) we obtain

$$\begin{aligned}
\frac{\partial f^{r+1}}{\partial x^i} &= \frac{\partial}{\partial x^i} \int_0^1 \sum_{j=1}^{p} x^j G_j(f^k, \lambda x^l)\,\mathrm{d}\lambda \\
&= \int_0^1 \sum_{j=1}^{p} x^j \lambda \frac{\partial G_j}{\partial x^i}(f^k, \lambda x^l) + G_i(f^k, \lambda x^l)\,\mathrm{d}\lambda \\
&= \int_0^1 \sum_{j=1}^{p} x^j \lambda \frac{\partial G_i}{\partial x^j}(f^k, \lambda x^l) + G_i(f^k, \lambda x^l)\,\mathrm{d}\lambda \\
&= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}\lambda}(\lambda G_i(f^k, \lambda x^l))\,\mathrm{d}\lambda = \lambda G_i(f^k, \lambda x^l)\Big|_0^1 = G_i(f^k, \lambda x^l)
\end{aligned} \tag{6}$$

for all $1 \le i \le p$. Thus, we have

$$\begin{aligned}
\mathrm{d}f^{r+1} &= \sum_{i=1}^{p} \frac{\partial f^{r+1}}{\partial x^i}\,\mathrm{d}x^i + \sum_{j=1}^{r} \frac{\partial f^{r+1}}{\partial f^j}\,\mathrm{d}f^j \\
&\equiv \sum_{i=1}^{p} G_i(f^k, x^l)\,\mathrm{d}x^i \mod \{\mathrm{d}f^1, \dots, \mathrm{d}f^r\}
\end{aligned} \tag{7}$$

and so from (3) we deduce $\omega_{r+1} \equiv \mathrm{d}f^{r+1} \mod \{\mathrm{d}f^1, \dots, \mathrm{d}f^r\}$. We conclude that

$$\{\omega_1, \dots, \omega_{r+1}\} = \{\mathrm{d}f^1, \dots, \mathrm{d}f^r, \omega_{r+1}\} = \{\mathrm{d}f^1, \dots, \mathrm{d}f^{r+1}\}\,.$$

By induction argument we can construct (using quadratures) the functions f^{r+2}, $\ldots$, f^{r+s}, such that $\{\omega_1, \ldots, \omega_{r+s}\} = \{\mathrm{d}f^1, \ldots, \mathrm{d}f^{r+s}\}$.

3. Lie Algebras of Symmetries and Integration by Quadratures

In this section we will explain how one can reduce a Frobenius system using a solvable Lie algebra of symmetries.

Theorem 1. *Let $\mathcal{I}$ be a rank $r \geq 1$ Frobenius system on a manifold M and let X be a non-trivial infinitesimal symmetry of $\mathcal{I}$. Denote*

$$\mathcal{I}_X = \{\omega \in \mathcal{I}; X \lrcorner\, \omega = 0\}$$

and let $\theta \in \mathcal{I}$ such that $X \lrcorner\, \theta \neq 0$. Then

$$\mathrm{d}\left(\frac{\theta}{X \lrcorner\, \theta}\right) \equiv 0 \mod \mathcal{I}_X \tag{8}$$

and $\mathcal{I}_X$ is a Frobenius system of rank $r-1$. Thus, $\mathcal{I}$ is reducible to $\mathcal{I}_X$. Moreover X is a trivial infinitesimal symmetry of $\mathcal{I}_X$.

Proof: Since $\theta \neq \mathcal{I}_X$ we write $\mathrm{d}\theta \equiv \alpha \wedge \theta \mod \mathcal{I}_X$, for some one-form α. Using Cartan's formula we deduce

$$\mathcal{L}_X\theta = X \lrcorner\ \mathrm{d}\theta + \mathrm{d}(X \lrcorner\, \theta) \equiv (X \lrcorner\, \alpha)\theta - (X \lrcorner\, \theta)\alpha + \mathrm{d}(X \lrcorner\, \theta) \mod \mathcal{I}_X\,. \tag{9}$$

Because $\mathcal{L}_X\theta \in \mathcal{I}$ we have $\mathcal{L}_X\theta \wedge \theta \equiv 0 \mod \mathcal{I}_X$, so wedging (9) with θ we obtain

$$\mathrm{d}(X \lrcorner\, \theta) \wedge \theta \equiv (X \lrcorner\, \theta)\alpha \wedge \theta \mod \mathcal{I}_X\,. \tag{10}$$

Next, we have

$$\mathrm{d}\left(\frac{\theta}{X \lrcorner\, \theta}\right) \equiv -\frac{1}{(X \lrcorner\, \theta)^2}\,\mathrm{d}(X \lrcorner\, \theta) \wedge \theta + \frac{1}{X \lrcorner\, \theta}\alpha \wedge \theta \mod \mathcal{I}_X\,. \tag{11}$$

Substituting (10) into (11) we conclude that (8) is satisfied. Let $\omega \in \mathcal{I}_X$. We have

$$X \lrcorner\ \mathrm{d}\omega = \mathcal{L}_X\omega \in \mathcal{I}\,. \tag{12}$$

We write

$$\mathrm{d}\omega \equiv \beta \wedge \theta \mod I_X\,. \tag{13}$$

Thus, we obtain

$$X \lrcorner\ \mathrm{d}\omega \equiv -(X \lrcorner\, \theta)\beta \mod \mathcal{I}\,. \tag{14}$$

From (12) and (14) follows $\beta \in \mathcal{I}$ and by (13)

$$\mathrm{d}\omega \equiv 0 \mod \mathcal{I}_X$$

and so $\mathcal{I}_X$ is Frobenius. By assumption $\mathcal{L}_X\omega \in \mathcal{I}$ and

$$X \lrcorner\, \mathcal{L}_X\omega = L_X(X \lrcorner\, \omega) - [X,X] \lrcorner\, \omega = 0$$

and so $L_X\omega \in \mathcal{I}_X$. This proves that X is a trivial infinitesimal symmetry of $\mathcal{I}_X$. □

We note, that if the Frobenius system $\mathcal{I} = \{\theta\}$ is of rank one and X is a non-trivial infinitesimal symmetry of $\mathcal{I}$, then the one-form

$$\bar{\theta} = \frac{\theta}{X \lrcorner\, \theta}$$

is closed, and so, by the of Poincaré lemma, $\bar{\theta}$ is locally exact.

Definition 2. *Lie algebra $\mathfrak{g}$ is called a solvable Lie algebra if there is a chain of Lie subalgebras*

$$\{0\} = \mathfrak{g}^{(0)} \subset \mathfrak{g}^{(1)} \subset \mathfrak{g}^{(2)} \subset \cdots \mathfrak{g}^{(s-1)} \subset \mathfrak{g}^{(s)} = \mathfrak{g}$$

such that for every $0 \le i \le s$, $\dim \mathfrak{g}^{(i)} = i$ and $\mathfrak{g}^{(i-1)}$ is a normal subalgebra (an ideal) of $\mathfrak{g}^{(i)}$, i. e.

$$[\mathfrak{g}^{(i-1)}, \mathfrak{g}^{(i)}] \subseteq \mathfrak{g}^{(i-1)}, \quad \textit{for} \quad 1 \le i \le s\,.$$

Theorem 2. *Let $\mathcal{I}$ be a Frobenius system of rank r and let $\mathfrak{g}$ be a solvable Lie algebra of infinitesimal symmetries of $\mathcal{I}$. Denote*

$$\mathcal{I}_{\mathfrak{g}} = \{\omega \in \mathcal{I};\, X \lrcorner\, \omega = 0,\ \textit{for all}\ X \in \mathfrak{g}\}\,.$$

Then $\mathcal{I}_{\mathfrak{g}}$ is a Frobenius system and $\mathcal{I}$ is reducible to $\mathcal{I}_{\mathfrak{g}}$. If $\mathcal{I}_{\mathfrak{g}} = \{0\}$, then $\mathcal{I}$ is solvable by quadratures.

Proof: Since $\mathfrak{g}$ is solvable then there is a basis $\{X_1, X_2, \ldots, X_s\}$ of $\mathfrak{g}$ such that

$$[X_i, X_j] = \sum_{k=1}^{j-1} c_{ij}^k X_k, \qquad \text{whenever} \qquad 1 \le i < j \le s$$

(note that c_{ij}^k are constants) and denote

$$\mathcal{I}_i = \{\omega \in \mathcal{I};\, X_j \lrcorner\, \omega = 0,\ \text{for all} 1 \le j \le i\} \quad \text{for}\ \ 1 \le i \le s\,.$$

To establish the theorem we prove that

i) $\mathcal{I}_i$ is a Frobenius system for $0 \le i \le s$, and
ii) X_{i+1} is an infinitesimal symmetry of $\mathcal{I}_i$ for $1 \le 1 \le s$.

Since $\mathcal{I}_0 = \mathcal{I}$ the conditions (i), (ii), are satisfied for $i = 0$. We now proceed by induction. Assume that (i), (ii), are satisfied for $i-1$. By definition $\mathcal{I}_\mathcal{I} = \{\omega \in \mathcal{I}_{i-1};\, X_\mathcal{I} \lrcorner\, \omega = 0\}$, and so, by Theorem 3.1, $\mathcal{I}_\mathcal{I}$ is Frobenius. We need to show that X_{i+1} is an infinitesimal symmetry of I_i. If $\omega \in \mathcal{I}_i$, then $\mathcal{L}_{X_{i+1}}\omega \in \mathcal{I}$ and for $1 \le j \le i$

$$X_j \lrcorner\, L_{X_{i+1}}\omega = \mathcal{L}_{X_{i+1}}(X_j \lrcorner\, \omega) - [X_{i+1}, X_j] \lrcorner\, \omega = 0\,.$$

Thus, $\mathcal{L}_{X_{i+1}}\omega \in \mathcal{I}_i$ and so $L_{X_{i+1}}\mathcal{I}_i \subseteq \mathcal{I}_i$. Now assume that rank of $\mathcal{I}_{i-1}$ is greater or equal then rank of $\mathcal{I}_\mathcal{I}$, i. e. there is $\theta^{i-1} \in \mathcal{I}_{i-1}$ such that $X_i \lrcorner \theta^{i-1} \neq 0$. By Theorem 3.1

$$\mathrm{d}\left(\frac{\theta^{i-1}}{X_i \lrcorner\, \theta^{i-1}}\right) \equiv 0 \mod \mathcal{I}_i$$

and so $\mathcal{I}_i$ is reducible to I_{i+1}. This ends the proof of the theorem. □

Note that in general, the search for a solvable s-dimensional Lie algebra of infinitesimal symmetries is as difficult as finding the family of s forms ω_{r+1}, $\ldots$, ω_{r+s} satisfying (1).

Example 1. *Consider the Darboux integrable equation at level one*

$$u_{xy} + \frac{2\sqrt{u_x u_y}}{x+y} = 0\,, \tag{15}$$

studied by Goursat [6] (see also Vessiot [13]). Together with the two compatible equations

$$\frac{u_{xx}}{2\sqrt{u_x}} + \frac{\sqrt{u_x}}{x+y} = f(x) \qquad \text{and} \qquad \frac{u_{yy}}{2\sqrt{u_y}} + \frac{\sqrt{u_y}}{x+y} = g(y)$$

where $f(x)$ and $g(y)$ are arbitrary functions, equation (15) *gives rise to the Frobenius system $\mathcal{I} = \{\theta^1, \theta^2, \theta^3\}$, where*

$$\theta^1 = \mathrm{d}u - u_x\,\mathrm{d}x - u_y\,\mathrm{d}y \tag{16}$$

$$\theta^2 = \mathrm{d}u_x + \Big(\frac{2u_x}{x+y} - 2\sqrt{u_x}f(x)\Big)\,\mathrm{d}x + \frac{2\sqrt{u_x u_y}}{x+y}\,\mathrm{d}y \tag{17}$$

$$\theta^3 = \mathrm{d}u_y + \frac{2\sqrt{u_x u_y}}{x+y}\,\mathrm{d}x + \Big(\frac{2u_y}{x+y} - 2\sqrt{u_y}g(y)\Big)\,\mathrm{d}y \tag{18}$$

on the manifold (x, y, u, u_x, u_y). The integral manifolds of (16), (17), (18) *on which $\mathrm{d}x \wedge \mathrm{d}y \neq 0$ are solutions to equation* (15). *Integrating the system* (16), (17), (18) *we obtain the general solution of* (15). *We first integrate the system $C(D_x) = \{\mathrm{d}y, \theta^1, \theta^2, \theta^3\}$, which is generated by*

$$\mathrm{d}y, \quad \omega^1 = \mathrm{d}u - u_x\,\mathrm{d}x$$

$$\omega^2 = \mathrm{d}u_x + (\frac{2u_x}{x+y} - 2\sqrt{u_x}f(x))\,\mathrm{d}x$$

$$\omega^3 = \frac{1}{2\sqrt{u_y}}\,\mathrm{d}u_y + \frac{\sqrt{u_x}}{x+y}\,\mathrm{d}x\,.$$

The structure equations for $C(D_x)$ *are* $\mathrm{d}\omega^1 = -\omega^2 \wedge \mathrm{d}x$,

$$\mathrm{d}\omega^2 \equiv \left(\frac{2}{x+y} - \frac{f(x)}{\sqrt{u_x}}\right)\omega^2 \wedge \mathrm{d}x \mod \mathrm{d}y$$

and

$$\mathrm{d}\omega^3 \equiv \frac{1}{2\sqrt{u_x}(x+y)}\omega^2 \wedge \mathrm{d}x \mod \mathrm{d}y\,.$$

It is not difficult to see that the vector field

$$v_1 = \frac{\sqrt{u_x}}{x+y}\frac{\partial}{\partial\omega^2}$$

is an infinitesimal symmetry of the Frobenius subsystem $\{\mathrm{d}y, \omega^2\}$ *of* $C(D_x)$. *By Theorem 3.1 the form*

$$\bar{\omega}^2 = \frac{1}{2}\frac{\omega^2}{X_1 \lrcorner\, \omega^2} \in \{\mathrm{d}y, \omega^2\}$$

is closed $\mod \mathrm{d}y$. *Indeed,* $\bar{\omega}^2 \equiv \mathrm{d}I_2 \mod \{\mathrm{d}y\}$, *where* $I_2 = (x+y)\sqrt{u_x} - (x+y)F'(x) + F(x)$, *and where* $F''(x) = f(x)$. *We have*

$$\omega^1 = \mathrm{d}u - \frac{(I_2 - F(x) + (x+y)F'(x))^2}{(x+y)^2}\,\mathrm{d}x$$

and

$$\omega^3 = \frac{1}{2\sqrt{u_y}}\,\mathrm{d}u_y + \frac{I_2 - F(x) + (x+y)F'(x)}{(x+y)^2}\,\mathrm{d}x.$$

and so it not difficult to conclude that

$$\omega^1 \equiv \mathrm{d}\left(u + \frac{(I_2 - F(x))^2}{x+y} - \int F'^2(x)\,\mathrm{d}x\right) \mod \{\mathrm{d}y, \mathrm{d}I_2\} \tag{19}$$

and

$$\omega^3 \equiv \mathrm{d}\left(\sqrt{u_y} + \frac{F(x) - I_2}{x+y}\right) \mod \{\mathrm{d}y, \mathrm{d}I_2\}\,. \tag{20}$$

Denote

$$I_1 = u + \frac{(I_2 - F(x))^2}{x+y} - \int F'^2(x)\,\mathrm{d}x \quad \text{and} \quad I_3 = \sqrt{u_y} + \frac{F(x) - I_2}{x+y}.$$

Hence, $C(D_x) = \{\mathrm{d}y, \mathrm{d}I_1, \mathrm{d}I_2, \mathrm{d}I_3\}$ *and Frobenius system* $\mathcal{I}$ *is generated by one-forms*

$$\eta^1 = \mathrm{d}I_1 - I_3^2\,\mathrm{d}y, \quad \eta^2 = \mathrm{d}I_2 + I_3\,\mathrm{d}y, \quad \textit{and} \quad \eta^3 = \mathrm{d}I_3 - g(y)\,\mathrm{d}y\,.$$

We have $\eta^3 = \mathrm{d}J_3$*, where* $J_3 = I_3 - G'(y)$ *and* $G''(y) = g(y)$*, and so*

$$\eta^1 = \mathrm{d}I_1 - (J_3 + G'(y))^2\,\mathrm{d}y \quad \textit{and} \quad \eta^2 = \mathrm{d}I_1 + (J_3 + G'(y))\,\mathrm{d}y$$

i. e.

$$\eta^1 \equiv \mathrm{d}\left(I_1 - yJ_3^2 - 2G(y)J_3 - \int G'^2(y)\,\mathrm{d}y\right) \bmod \{\mathrm{d}J_3\} \tag{21}$$

and

$$\eta^2 \equiv \mathrm{d}(I_2 + yJ_3 + G(y)) \bmod \{\mathrm{d}J_3\}\,. \tag{22}$$

Denote

$$J_1 = I_1 - yJ_3^2 - 2G(y)J_3 - \int G'^2(y)\,\mathrm{d}y \quad \textit{and} \quad J_2 = I_2 + yJ_3 + G(y)\,.$$

Thus, $\mathcal{I} = \{\mathrm{d}J_1, \mathrm{d}J_2, \mathrm{d}J_3\}$*, and setting* $J_1 = J_2 = J_3 = 0$ *we obtain*

$$I_1 = \int G'^2(y)\,\mathrm{d}y, \qquad I_2 = -G(y), \qquad \textit{and} \qquad I_3 = G'(y)\,.$$

From these equations we solve for u*,* u_x*, and* u_y *in terms of* x *and* y*. In particular we get*

$$u = -\frac{(F(x) + G(y))^2}{x + y} + \int F'^2(x)\,\mathrm{d}x + \int G'^2(y)\,\mathrm{d}y$$

which is, as one can easily verify, the general solution to (15*).*

4. Frobenius Systems of Even Rank

In this section we extend Noether's theorem for ordinary differential equations to Frobenius systems. As a consequence we obtain more powerful Lie reduction method when dealing with a solvable group of variational symmetries. Exact relationship between the solvable group of variational symmetries of $\mathcal{I}$ and the rank of the reduced system will be established.

Recall that a closed two-form ω has *rank* r if $\omega^r \neq 0$ and $\omega^{r+1} = 0$ ($\omega^r = \omega \wedge \cdots \wedge \omega$ r-times). For a Frobenius system $\mathcal{I}$ of even rank $2k$ we can always find a closed two-form Π in $\mathcal{I} \wedge \mathcal{I}$ of rank k. To exhibit this consider the local coordinates on the underlying manifold M guaranteed by

Frobenius theorem, $(p^1, q^1 \dots, p^k, q^k, x^1 \dots, x^s)$, such that $\mathcal{I}$ is generated by $\{dp^1, dq^1 \dots, dp^k, dq^k\}$. We now set

$$\Pi = dp^1 \wedge dq^1 + \cdots + dp^k \wedge dq^k .$$

A vector field X on M is called an **infinitesimal symmetry** of Π if $\mathcal{L}_X \Pi = 0$. Since $\mathcal{L}_{[X,Y]}\omega = L_X \mathcal{L}_Y \omega - \mathcal{L}_Y \mathcal{L}_X \omega$, we easily observe that the set of all infinitesimal symmetries of Π forms a Lie algebra.

Lemma 1. *Let $\mathcal{I}$ be a rank $2k$ Frobenius system on a manifold M and let $\Pi \in \mathcal{I} \wedge \mathcal{I}$ be a closed two-form of rank k. If a vector field X on M is an infinitesimal symmetry of Π, i. e. $\mathcal{L}_X \Pi = 0$, then X is also an infinitesimal symmetry of $\mathcal{I}$.*

Proof: It suffices to work locally. Let $m \in M$. By theorems of Darboux and Frobenius we deduce there is a neighborhood $U \subset M$ around m such that on U the system $\mathcal{I}$ is generated by $\{dp^1, dq^1, \dots, dp^k, dq^k\}$ and

$$\Pi = dp^1 \wedge dq^1 + \cdots + dp^k \wedge dq^k . \tag{23}$$

Let $\mathcal{L}_X \Pi = 0$, then

$$0 = \mathcal{L}_X \Pi = \sum_{i=1}^{k} (\mathcal{L}_X \, dp^i \wedge dq^i + dp^i \wedge \mathcal{L}_X \, dq^i)$$

and so it follows that $\mathcal{L}_X \, dp^i \equiv \mathcal{L}_X \, dq^i \equiv 0 \mod \mathcal{I}$. □

Lemma 2. *Let $\mathcal{I}$ be a rank $2k$ Frobenius system on M and let $\Pi \in \mathcal{I} \wedge \mathcal{I}$ be a closed two-form of rank k. If X is a vector field on M, then $X \lrcorner \Pi = 0$ if and only if X is a trivial symmetry of $\mathcal{I}$.*

Proof: It suffices to prove the Lemma locally. Applying first the Theorem of Frobenius and then the Theorem of Darboux we deduce that there are local coordinates on M, $(p^1, q^1 \dots, p^k, q^k, x^1 \dots, x^t)$, $(t = \dim M - 2k)$ such that $\mathcal{I} = \{dp^1, dq^1, \dots, dp^k, dq^k\}$ and Π is given by (23). Let

$$X = \sum_{i=1}^{k} a_i \frac{\partial}{\partial p^i} + \sum_{i=1}^{k} b_i \frac{\partial}{\partial q^i} + \sum_{i=1}^{t} c_i \frac{\partial}{\partial x^i} .$$

Then

$$X \lrcorner \Pi = \sum_{i=1}^{k} a_i \, dq^i - \sum_{i=1}^{k} b_i \, dp^i = 0 \quad \text{iff} \quad a_i = b_i = 0 \quad \text{for all } \; i = 1, \dots, k.$$

This is equivalent to $X \lrcorner \, dp^i = 0$ and $X \lrcorner \, dq^i = 0$ for all $i = 1, \dots, k$. □

Theorem 3. *Let $\mathcal{I}$ be a rank $2k$ Frobenius system on a manifold M and let $\Pi \in \mathcal{I} \wedge \mathcal{I}$ be a closed two-form of rank k. Then there is locally a one-to-one correspondence between the equivalence classes of infinitesimal symmetries of Π and the equivalence classes of conservation laws of $\mathcal{I}$.*

Proof: As before assume that on some neighborhood U of $M, \mathcal{I}$ is generated by $\{\mathrm{d}p^1, \mathrm{d}q^1, \ldots, \mathrm{d}p^k, \mathrm{d}q^k\}$ and Π is given by (23). From now on we restrict our considerations to this neighborhood U. Let $\mathrm{Sym}(\Pi)$ be the vector space over real numbers of all infinitesimal symmetries of Π and let $\mathrm{CL}(\mathcal{I})$ be the real vector space of all conservation laws. Let us define an equivalence relation $\sim$ on $\mathrm{Sym}(\Pi)$. For $X_1, X_2 \in \mathrm{Sym}(\Pi)$ we define $X_1 \sim X_2$ if and only if $X_1 - X_2$ is a trivial symmetry. Define an equivalence relation $\sim$ on $\mathrm{CL}(\mathcal{I})$: for $f^1, f^2 \in \mathrm{CL}(\mathcal{I})$ we define $f^1 \sim f^2$ if and only if $f^1 - f^2$ is a constant. Define a map ϕ: $\mathrm{Sym}(\Pi)/\!\sim\, \to \mathrm{CL}(\mathcal{I})/\!\sim$ as follows: let $\bar{X} \in \mathrm{Sym}(\Pi)/\!\sim$ and let $X \in \bar{X}$. By Cartan's formula we have

$$\mathrm{d}(X \lrcorner\, \Pi) = X \lrcorner\, \mathrm{d}\Pi + \mathrm{d}(X \lrcorner\, \Pi) = \mathcal{L}_X \Pi = 0\,.$$

By Poincaré Lemma, there is a function f such that $\mathrm{d}f = X \lrcorner\, \Pi \in \mathcal{I}$ and if $\mathrm{d}f = \mathrm{d}g = X \lrcorner\, \Pi$, then $f - g = \mathrm{const}$. Thus, f is a conservation law of $\mathcal{I}$ and we set

$$\phi(\bar{X}) = \bar{f} = \{f + c;\ c \text{ is a real number}\}\,.$$

By Lemma 4.2 ϕ is an injection. To prove that ϕ is a surjection we consider a local conservation law f, i. e. a smooth function on $U \subseteq M$ such that $\mathrm{d}f = \sum_{i=1}^{k}(a_i\, \mathrm{d}p^i + b_i\, \mathrm{d}q^i)$ for some functions a_i and b_i. Let

$$X = \sum_{i=1}^{k} \left(-a_i \frac{\partial}{\partial q^i} + b_i \frac{\partial}{\partial p^i} \right).$$

Then $X \lrcorner\, \Pi = \sum_{i=1}^{k}(a_i\, \mathrm{d}p^i + b_i\, \mathrm{d}q^i) = \mathrm{d}f$ and so $\phi(\bar{X}) = \bar{f}$. Here again $\bar{X} \in \mathrm{Sym}(\Pi)/\!\sim$ such that $X \in \bar{X}$. □

This Theorem generalizes a special case of Noether's Theorem for ordinary differential equations and is not, to my knowledge, included in the most general statement of Noether's theorem. Recall that Noether's theorem gives a one-to-one correspondence between variational (or divergence) symmetries of a non-degenerate Lagrangian and conservation laws of the associated system of Euler–Lagrange equations. If the Lagrangian L of order k depends only on one independent variable, the corresponding system of Euler–Lagrange equations can be interpreted in terms of differential forms as a Frogenius system of even rank $2k$. A variational symmetry of a Lagrangian L is a vector field X whose Lie derivative carries the Poincaré–Cartan form η_L into an exact one-form, i. e. $\mathcal{L}_X \eta_L = \mathrm{d}f$, for some function f. Locally this is equivalent to saying that $L_X\, \mathrm{d}\eta_L = 0$ and one can easily show that $\Pi = \mathrm{d}\eta_L$ is a closed two-form in $\mathcal{I} \wedge \mathcal{I}$ of rank k. The minor novelty of Theorem 4.3 lies in the fact that Π does not have to arise from a variational principle. Note that Lemma 4.1 is a generalization of a well known fact that every infinitesimal divergence symmetry of a Lagrangian is also a symmetry of its Euler–Lagrange equations.

It is well known that a one-parameter group G of variational symmetries allows one to reduce the order of the associated Euler–Lagrange equations by two. What happens when G is an s- parameter solvable group of variational symmetries is the topic of our further investigations.

As before, let $\mathcal{I}$ be a Frobenius system of rank $2k$ and let $\mathfrak{g}$ be a Lie algebra of infinitesimal symmetries of a closed two-form $\Pi \in I \wedge \mathcal{I}$ of rank k. By Lemma 4.1 $\mathfrak{g}$ is also a Lie algebra of infinitesimal symmetries of $\mathcal{I}$. Thus, $\mathcal{I}_{\mathfrak{g}} = \{\omega \in \mathcal{I};\ X \lrcorner\, \omega = 0 \text{ for all } X \in \mathfrak{g}\}$ is a Frobenius system. It is not hard to see that the set

$$\Pi_{\mathfrak{g}} = \{X \lrcorner\, \Pi;\ X \in \mathfrak{g}\}$$

is a real vector space with generators $\{X_i \lrcorner\, \Pi\}$, where $\{X_i\}$ is a basis of $\mathfrak{g}$. Using Cartan's formula, for $X \in \mathfrak{g}$, follows $\mathrm{d}(X \lrcorner\, \Pi) = \mathcal{L}_X \Pi = 0$, and so every form in $\Pi_{\mathfrak{g}}$ is closed. Therefore, locally we have $t = \dim \Pi_{\mathfrak{g}}$ functionally independent conservation laws. If, moreover,

$$Y \lrcorner\, X \lrcorner\, \Pi = 0, \qquad \text{for all} \quad X, Y \in \mathfrak{g} \tag{24}$$

then $\Pi_{\mathfrak{g}} \subseteq \mathcal{I}_{\mathfrak{g}}$.

We can now combine Theorem 3.3 with the above observation to obtain an integration method that essentially doubles the power of the standard Lie symmetry method.

Theorem 4. *Let $\mathcal{I}$ be a rank $2k$ Frobenius system on a manifold M and $\Pi \in \mathcal{I} \wedge \mathcal{I}$ be a closed two-form of rank k. Let $\mathfrak{g}$ be a solvable Lie algebra of infinitesimal symmetries of Π and let*

$$r = \operatorname{rank} \mathcal{I}_{\mathfrak{g}} - \dim(\Pi_{\mathfrak{g}} \cap \mathcal{I}_{\mathfrak{g}})\,.$$

Then $\mathcal{I}$ is reducible to a Frobenius system of rank r.

Proof: By Theorem 3.3 $\mathcal{I}$ is reducible to $\mathcal{I}_{\mathfrak{g}}$. Let t be the dimension of $\Pi_{\mathfrak{g}} \cap \mathcal{I}_{\mathfrak{g}}$ (as a real vector space). Then there are t linearly independent closed forms that form a basis of $\Pi_{\mathfrak{g}} \cap I_{\mathfrak{g}}$. Therefore $\mathcal{I}_{\mathfrak{g}}$ can be further reduced using these closed forms. □

Now consider a special type of solvable Lie algebras $\mathfrak{g}$ for which condition (24) is satisfied.

Theorem 5. *Let $\mathcal{I}$ be a rank $2k$ Frobenius system on a manifold M and $\Pi \in \mathcal{I} \wedge \mathcal{I}$ be a closed two-form of rank k. Let $\mathfrak{g}$ be a solvable Lie algebra of infinitesimal symmetries of Π of dimension s such that the condition (24) is satisfied. Let*

$$r = \operatorname{rank} \mathcal{I}_{\mathfrak{g}} - \dim \Pi_{\mathfrak{g}} .$$

Then $\mathcal{I}$ is reducible to a Frobenius system of rank r. In particular if $\operatorname{rank} \mathcal{I}_{\mathfrak{g}} = \dim \Pi_{\mathfrak{g}}$, *then $\mathcal{I}$ is integrable by quadratures. In that case it is necessary that $s \geq k$.*

Proof: Follows from the previous theorem and the fact that when condition (24) is satisfied, then $\Pi_{\mathfrak{g}} \subseteq \mathcal{I}_{\mathfrak{g}}$. To prove the last claim assume that $s < k$ and so

$$\operatorname{rank} \mathcal{I}_{\mathfrak{g}} \geq 2k - s > k > s \geq \Pi_{\mathfrak{g}}$$

which is a contradiction. □

Corollary 1. *Let $\mathcal{I}$ be a rank $2k$ Frobenius system on a manifold M and $\Pi \in \mathcal{I} \wedge \mathcal{I}$ be a closed two-form of rank k. Let X be an infinitesimal symmetry of Π. Then $\mathcal{I}$ is reducible to a Frobenius system of rank $2k - 2$.*

Proof: $X \lrcorner X \lrcorner \Pi = 0$ and so the condition (24) is satisfied. $\operatorname{rank} \mathcal{I}_{\mathfrak{g}} = 2k - 1$ and $\dim \Pi_{\mathfrak{g}} = 1$ and so the statement folows from Theorem 4.5. □

This Corollary generalizes Theorem 4.17 in [11], page 258.

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