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AN INTRODUCTION TO MOVING FRAMES

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Abstract. This paper surveys the new, algorithmic theory of moving frames. Applications in geometry, computer vision, classical invariant theory, and numerical analysis are indicated.

1. Introduction

The method of moving frames ("repères mobiles") was forged by Élie Cartan, [7, 8], into a powerful and algorithmic tool for studying the geometric properties of submanifolds and their invariants under the action of a transformation group. However, Cartan's methods remained incompletely understood and the applications were exclusively concentrated in classical differential geometry, see [12, 13, 15]. In the late 1990's, we have formulated in [10, 11] a new approach to the moving frame theory that can be systematically applied to general transformation groups. The key idea is to define a moving frame as an equivariant map to the transformation group. All classical moving frames can be reinterpreted in this manner, but the new approach applies in far wider generality. Cartan's construction of the moving frame through the normalization process corresponds to the choice of a cross-section to the group orbits. Building on these two simple ideas, one may algorithmically construct moving frames and complete systems of invariants for completely general group actions. The existence of a moving frame requires freeness of the underlying group action.

Classically, non-free actions are made free by prolonging to jet space, leading to differential invariants and the solution to equivalence and symmetry problems via the differential invariant signature. More recently, the moving frame method was also applied to Cartesian product actions, leading to classification of joint invariants and joint differential invariants, [26]. The combination of jet and Cartesian product actions known as multi-space was proposed in [27] as a framework for the geometric analysis of numerical approximations, and, via the application of the moving frame method, to the systematic construction of invariant numerical algorithms.

New and significant applications of these results have been developed in a wide variety of directions. In [1, 17, 24], the theory was applied to produce new algorithms for solving the basic symmetry and equivalence problems of polynomials that form the foundation of classical invariant theory. In [19], the differential invariants of projective surfaces were classified and applied to generate integrable Poisson flows arising in soliton theory. Faugeras [9], initiated the applications of moving frames in computer vision, and in [6], the characterization of submanifolds via their differential invariant signatures was applied to the problem of object recognition and symmetry detection in digital images. The moving frame method provides a direct route to the classification of joint invariants and joint differential invariants [11, 26], establishing a geometric counterpart of what Weyl [30], in the algebraic framework, calls the first main theorem for the transformation group. The approximation of higher order differential invariants by joint differential invariants and, generally, ordinary joint invariants leads to fully invariant finite difference numerical schemes, first proposed in [3, 5, 6, 27]. Moving frames have been used to find a complete solution to the calculus of variations problem of directly constructing differential invariant Euler-Lagrange equations from their differential invariant Lagrangians [18]. Finally, the theory has recently been extended to the vastly more complicated arena of infinite-dimensional Lie pseudo-groups [28, 29].

2. Moving Frames

We begin by outlining the basic moving frame construction in [11]. Let G be an r-dimensional Lie group acting smoothly on an m-dimensional manifold M. Let $G_S = \{g \in G;\ g \cdot S = S\}$ denote the **isotropy subgroup** of a subset $S \subset M$, and $G_S^* = \bigcap_{z \in S} G_z$ its **global isotropy subgroup**, which consists of those group elements which fix all points in S. We always assume, without any significant loss of generality, that G acts *effectively on subsets*, and so $G_U^* = \{e\}$ for any open $U \subset M$, i.e. there are no group elements other than the identity which act completely trivially on an open subset of M.

The crucial idea is to decouple the moving frame theory from reliance on any form of frame bundle. In other words,

$$\text{Moving frames} \neq \text{Frames!}$$

A careful study of Cartan's analysis of the case of projective curves [7], reveals that Cartan was well aware of this fact. However, this important and instructive example did not receive the attention it deserves.

Definition 1. A **moving frame** is a smooth, G-equivariant map $\rho : M \to G$.

The group G acts on itself by left or right multiplication. If $\rho(z)$ is any right-equivariant moving frame then $\widetilde{\rho}(z) = \rho(z)^{-1}$ is left-equivariant and conversely. All classical moving frames are left-equivariant, but, in many cases, the right versions are easier to compute. In many geometrical situations, one can identify our left moving frames with the usual frame-based versions, but these identifications break down for more general transformation groups.

Theorem 2. *A moving frame exists in a neighborhood of a point $z \in M$ if and only if G acts freely and regularly near z.*

Recall that G acts *freely* if the isotropy subgroup of each point is trivial, $G_z = \{e\}$ for all $z \in M$. This implies that the orbits all have the same dimension as G itself. *Regularity* requires that, in addition, each point $x \in M$ has a system of arbitrarily small neighborhoods whose intersection with each orbit is connected, cf. [22].

The practical construction of a moving frame is based on Cartan's method of *normalization* [16, 7], which requires the choice of a (local) *cross-section* to the group orbits.

Theorem 3. *Let G act freely and regularly on M, and let $K \subset M$ be a cross-section. Given $z \in M$, let $g = \rho(z)$ be the unique group element that maps z to the cross-section: $g \cdot z = \rho(z) \cdot z \in K$. Then $\rho : M \to G$ is a right moving frame for the group action.*

Given local coordinates $z = (z_1, \ldots, z_m)$ on M, let $w(g, z) = g \cdot z$ be the explicit formulae for the group transformations. The right moving frame $g = \rho(z)$ associated with a *coordinate cross-section* $K = \{ z_1 = c_1, \ldots, z_r = c_r \}$ is obtained by solving the *normalization equations*

$$w_1(g, z) = c_1, \qquad \ldots \qquad w_r(g, z) = c_r \tag{1}$$

for the group parameters $g = (g_1, \ldots, g_r)$ in terms of the coordinates $z = (z_1, \ldots, z_m)$. Substituting the moving frame formulae into the remaining transformation rules leads to a complete system of invariants for the group action.

Theorem 4. *If $g = \rho(z)$ is the moving frame solution to the normalization equations* (1)*, then the functions*

$$I_1(z) = w_{r+1}(\rho(z), z), \qquad \ldots \qquad I_{m-r}(z) = w_m(\rho(z), z) \tag{2}$$

form a complete system of functionally independent invariants.

Definition 5. The **invariantization** of a scalar function $F : M \to \mathbb{R}$ with respect to a right moving frame ρ is the the invariant function $I = \iota(F)$ defined by $I(z) = F(\rho(z) \cdot z)$.

Invariantization amounts to restricting F to the cross-section, $I_{|K} = F_{|K}$, and then requiring that I be constant along the orbits. In particular, if $I(z)$ is an invariant, then $\iota(I) = I$, so invariantization defines a projection, depending on the moving frame, from functions to invariants. Thus, a moving frame provides a canonical method of associating an invariant with an arbitrary function and more generally [18], invariant differential forms with ordinary differential forms.

Of course, most interesting group actions are *not* free, and therefore do not admit moving frames in the sense of Definition 1. There are two common methods for converting a non-free (but effective) action into a free action. In the traditional moving frame theory [7, 13, 15], this is accomplished by prolonging the action to a jet space J^n of suitably high order and the consequential invariants are the classical differential invariants for the group [11, 22]. Alternatively, one may consider the product action of G on a sufficiently large Cartesian product $M^{\times(n+1)}$. Here, the invariants are joint invariants [26], of particular interest in classical algebra [24, 30]. In neither case is there a general theorem guaranteeing the freeness and regularity of the prolonged or product actions, (indeed, there are counterexamples in the product case), but such pathologies never occur in practical examples. In our approach to invariant numerical approximations [27], the two methods are amalgamated by prolonging to an appropriate multi-space.

3. Prolongation and Differential Invariants

Traditional moving frames are obtained by prolonging the group action to the n-th order (extended) jet bundle $\mathrm{J}^n = \mathrm{J}^n(M, p)$ consisting of equivalence classes of p-dimensional submanifolds $S \subset M$ modulo n-th order contact at a single point, see [22, Chapter 3] for details. Since G preserves the contact equivalence relation, it induces an action on the jet space J^n, known as its n-th order **prolongation** and denoted by $G^{(n)}$.

An ***n*-th order moving frame** $\rho^{(n)} : \mathrm{J}^n \to G$ is an equivariant map defined on an open subset of the jet space. In practical examples, for n sufficiently large, the prolonged action $G^{(n)}$ becomes regular and free on a dense open subset $\mathcal{V}^n \subset \mathrm{J}^n$, the set of *regular jets*.

Theorem 6. *An n-th order moving frame exists in a neighborhood of a point $z^{(n)} \in \mathrm{J}^n$ if and only if $z^{(n)} \in \mathcal{V}^n$ is a regular jet.*

The normalization construction will produce a moving frame and a complete system of differential invariants in the neighborhood of any regular jet. Local coordinates $z = (x, u)$ on M – considering the first p components $x = (x^1, \dots, x^p)$ as independent variables, and the latter $q = m - p$ components $u = (u^1, \dots, u^q)$ as dependent variables – induce local coordinates $z^{(n)} = (x, u^{(n)})$ on J^n with components u_J^α representing the partial derivatives of the dependent variables with respect to the independent variables [22, 23]. We compute the prolonged transformation formulae

$$w^{(n)}(g, z^{(n)}) = g^{(n)} \cdot z^{(n)}, \qquad \text{or} \qquad (y, v^{(n)}) = g^{(n)} \cdot (x, u^{(n)})$$

by implicit differentiation of the v's with respect to the y's. For simplicity, we restrict to a coordinate cross-section by choosing $r = \dim G$ components of $w^{(n)}$ to normalize to constants:

$$w_1(g, z^{(n)}) = c_1, \qquad \ldots \qquad w_r(g, z^{(n)}) = c_r. \tag{3}$$

Solving the normalization equations (3) for the group transformations leads to the explicit formulae $g = \rho^{(n)}(z^{(n)})$ for the right moving frame. As in Theorem 4, substituting the moving frame formulae into the unnormalized components of $w^{(n)}$ leads to the *fundamental n-th order differential invariants*

$$I^{(n)}(z^{(n)}) = w^{(n)}(\rho^{(n)}(z^{(n)}), z^{(n)}) = \rho^{(n)}(z^{(n)}) \cdot z^{(n)}. \tag{4}$$

Once the moving frame is established, the *invariantization* process will map general differential functions $F(x, u^{(n)})$ to differential invariants $I = \iota(F) = F \circ I^{(n)}$. As before, invariantization defines a projection, depending on the moving frame, from the space of differential functions to the space of differential invariants. The fundamental differential invariants $I^{(n)}$ are obtained by invariantization of the coordinate functions

$$\begin{aligned} H^i(x, u^{(n)}) &= \iota(x^i) = y^i(\rho^{(n)}(x, u^{(n)}), x, u) \\ I^\alpha_K(x, u^{(k)}) &= \iota(u^\alpha_J) = v^\alpha_K(\rho^{(n)}(x, u^{(n)}), x, u^{(k)}). \end{aligned} \tag{5}$$

In particular, those corresponding to the normalization components (3) of $w^{(n)}$ will be constant, and are known as the *phantom differential invariants*.

Theorem 7. *Let $\rho^{(n)} : \mathrm{J}^n \to G$ be a moving frame of order $\leq n$. Every n-th order differential invariant can be locally written as a function $J = \Phi(I^{(n)})$ of the fundamental n-th order differential invariants* (5)*. The function Φ is unique provided it does not depend on the phantom invariants.*

Example 8. Let us illustrate the theory with a very simple, well-known example: curves in the Euclidean plane. The orientation-preserving Euclidean group SE(2) acts on $M = \mathbb{R}^2$, mapping a point $z = (x, u)$ to

$$y = x\cos\theta - u\sin\theta + a, \qquad v = x\sin\theta + u\cos\theta + b. \tag{6}$$

For a general parametrized curve $z(t) = (x(t), u(t))$, the prolonged group transformations

$$v_y = \frac{\mathrm{d}v}{\mathrm{d}y} = \frac{\dot{x}\sin\theta + \dot{u}\cos\theta}{\dot{x}\cos\theta - \dot{u}\sin\theta}, \qquad v_{yy} = \frac{\mathrm{d}^2 v}{\mathrm{d}y^2} = \frac{\dot{x}\ddot{u} - \ddot{x}\dot{u}}{(\dot{x}\cos\theta - \dot{u}\sin\theta)^3} \tag{7}$$

and so on, are found by successively applying the implicit differentiation operator

$$\frac{\mathrm{d}}{\mathrm{d}y} = \frac{1}{\dot{x}\cos\theta - \dot{u}\sin\theta}\,\frac{\mathrm{d}}{\mathrm{d}t} \tag{8}$$

to v. The classical Euclidean moving frame for planar curves [13], follows from the cross-section normalizations

$$y = 0, \qquad v = 0, \qquad v_y = 0. \tag{9}$$

Solving for the group parameters $g = (\theta, a, b)$ leads to the right-equivariant moving frame

$$\theta = -\tan^{-1}\frac{\dot{u}}{\dot{x}}, \qquad a = -\frac{x\dot{x} + u\dot{u}}{\sqrt{\dot{x}^2 + \dot{u}^2}} = \frac{z\cdot\dot{z}}{\|\dot{z}\|}, \qquad b = \frac{x\dot{u} - u\dot{x}}{\sqrt{\dot{x}^2 + \dot{u}^2}} = \frac{z\wedge\dot{z}}{\|\dot{z}\|}. \tag{10}$$

The inverse group transformation $g^{-1} = (\widetilde{\theta}, \widetilde{a}, \widetilde{b})$ is the classical left moving frame [7, 13]. One identifies the translation component $(\widetilde{a}, \widetilde{b}) = (x, u) = z$ as the point on the curve, while the columns of the rotation matrix $R(\widetilde{\theta}) = (\mathbf{t}, \mathbf{n})$ are the unit tangent and unit normal vectors. Substituting the moving frame normalizations (10) into the prolonged transformation formulae (7), results in the fundamental differential invariants

$$\begin{aligned} &v_{yy} \longmapsto \kappa = \frac{\dot{x}\ddot{u} - \ddot{x}\dot{u}}{(\dot{x}^2 + \dot{u}^2)^{3/2}} = \frac{\dot{z}\wedge\ddot{z}}{\|\dot{x}\|^3} \\ &v_{yyy} \longmapsto \frac{\mathrm{d}\kappa}{\mathrm{d}s}, \qquad v_{yyyy} \longmapsto \frac{\mathrm{d}^2\kappa}{\mathrm{d}s^2} + 3\kappa^3 \end{aligned} \tag{11}$$

where $\mathrm{d}/\mathrm{d}s = \|\dot{z}\|^{-1}\,\mathrm{d}/\mathrm{d}t$ is the arc length derivative – which is itself found by substituting the moving frame formulae (10) into the implicit differentiation operator (8). A complete system of differential invariants for the planar Euclidean group is provided by the curvature and its successive derivatives with respect to arc length: $\kappa, \kappa_s, \kappa_{ss}, \ldots$.

Example 9. Let $n \neq 0, 1$. In classical invariant theory, the planar actions

$$y = \frac{\alpha x + \beta}{\gamma x + \delta}, \qquad \overline{u} = (\gamma x + \delta)^{-n} u \tag{12}$$

of $G = \mathrm{GL}(2)$ play a key role in the equivalence and symmetry properties of binary forms, when $u = q(x)$ is a polynomial of degree $\leq n$, [14, 24, 1]. We identify the graph of the function $u = q(x)$ as a plane curve. The prolonged action on such graphs is found by implicit differentiation. Setting $\sigma = \gamma p + \delta$, $\Delta = \alpha\delta - \beta\gamma \neq 0$, we find

$$v_y = \frac{\sigma u_x - n\gamma u}{\Delta\sigma^{n-1}}, \qquad v_{yy} = \frac{\sigma^2 u_{xx} - 2(n-1)\gamma\sigma u_x + n(n-1)\gamma^2 u}{\Delta^2\sigma^{n-2}}$$

$$v_{yyy} = \frac{\sigma^3 u_{xxx} - 3(n-2)\gamma\sigma^2 u_{xx} + 3(n-1)(n-2)\gamma^2\sigma u_x - n(n-1)(n-2)\gamma^3 u}{\Delta^3\sigma^{n-3}}$$

and so on. On the sub-domain

$$\mathcal{V}^2 = \{H \neq 0\} \subset \mathrm{J}^2, \quad \text{where} \quad H = u u_{xx} - \frac{n-1}{n} u_x^2$$

is the classical Hessian covariant of u, we can choose the cross-section defined by the normalizations

$$y = 0, \qquad v = 1, \qquad v_y = 0, \qquad v_{yy} = 1.$$

Solving for the group parameters gives the right moving frame formulae

$$\begin{aligned} \alpha &= u^{(1-n)/n}\sqrt{H}, & \beta &= -x u^{(1-n)/n}\sqrt{H} \\ \gamma &= \tfrac{1}{n} u^{(1-n)/n} u_x, & \delta &= u^{1/n} - \tfrac{1}{n} x u^{(1-n)/n} u_x. \end{aligned} \tag{13}$$

Substituting the normalizations (13) into the higher order transformation rules gives us the differential invariants, the first two of which are

$$v_{yyy} \longmapsto J = \frac{T}{H^{3/2}}, \qquad v_{yyyy} \longmapsto K = \frac{V}{H^2} \tag{14}$$

where

$$T = u^2 u_{xxx} - 3\frac{n-2}{n} u u_x u_{xx} + 2\frac{(n-1)(n-2)}{n^2} u_x^3, \qquad V = u^3 u_{xxxx}$$
$$- 4\frac{n-3}{n} u^2 u_x u_{xx} + 6\frac{(n-2)(n-3)}{n^2} u u_x^2 u_{xx} - 3\frac{(n-1)(n-2)(n-3)}{n^3} u_x^4$$

and can be identified with classical covariants, which may be constructed using the basic transvectant process of classical invariant theory, cf. [14, 24]. Using $J^2 = T^2/H^3$ as the fundamental differential invariant will remove the ambiguity caused by the square root. As in the Euclidean case, higher order differential invariants are found by successive application of the normalized implicit differentiation operator $D_s = u H^{-1/2} D_x$ to the fundamental invariant J.

4. Equivalence and Signatures

The moving frame method was developed by Cartan expressly for the solution to problems of equivalence and symmetry of submanifolds under group actions. Two submanifolds $S, \bar{S} \subset M$ are said to be *equivalent* if $\bar{S} = g \cdot S$ for some $g \in G$. A *symmetry* of a submanifold is a group transformation that maps S to itself, and so is an element $g \in G_S$. As emphasized by Cartan [7], the solution to the equivalence and symmetry problems for submanifolds is based on the functional interrelationships among the fundamental differential invariants restricted to the submanifold.

Suppose we have constructed an n-th order moving frame $\rho^{(n)} : \mathrm{J}^n \to G$ defined on an open subset of jet space. A submanifold S is called *regular* if its n-jet $\mathrm{j}_n S$ lies in the domain of definition of the moving frame. For any $k \geq n$, we use $J^{(k)} = I^{(k)} \mid S = I^{(k)} \circ \mathrm{j}_k S$ to denote the k-th order *restricted differential invariants*. The k-th order *signature* $S^{(k)} = S^{(k)}(S)$ is the set parametrized by the restricted differential invariants; S is called *fully regular* if $J^{(k)}$ has constant rank $0 \leq t_k \leq p = \dim S$ for all $k \geq n$. In this case, $S^{(k)}$ forms a submanifold of dimension t_k – perhaps with self-intersections. In the fully regular case, $t_n < t_{n+1} < t_{n+2} < \cdots < t_s = t_{s+1} = \cdots = t \leq p$, where t is the *differential invariant rank* and s the *differential invariant order* of S.

Theorem 10. *Two fully regular p-dimensional submanifolds $S, \bar{S} \subset M$ are (locally) equivalent, $\bar{S} = g \cdot S$, if and only if they have the same differential invariant order s and their signature manifolds of order $s+1$ are identical: $S^{(s+1)}(\bar{S}) = S^{(s+1)}(S)$.*

Since symmetries are the same as self-equivalences, the signature also determines the symmetry group of the submanifold.

Theorem 11. *If $S \subset M$ is a fully regular p-dimensional submanifold of differential invariant rank t, then its symmetry group G_S is an $(r - t)$-dimensional subgroup of G that acts locally freely on S.*

A submanifold with maximal differential invariant rank $t = p$, and hence only a discrete symmetry group, is called *nonsingular*. The number of symmetries is determined by the **index** of the submanifold, defined as the number of points in S map to a single generic point of its signature:

$$\operatorname{ind} S = \min\left\{ \# \ (J^{(s+1)})^{-1}\{\zeta\} \, ; \ \zeta \in S^{(s+1)} \right\}.$$

Theorem 12. *If S is a nonsingular submanifold, then its symmetry group is a discrete subgroup of cardinality $\# G_S = \operatorname{ind} S$.*

At the other extreme, a rank 0 or *maximally symmetric* submanifold has all constant differential invariants, and so its signature degenerates to a single point.

Theorem 13. *A regular p-dimensional submanifold S has differential invariant rank 0 if and only if it is the orbit, $S = H \cdot z_0$, of a p-dimensional subgroup $H = G_S \subset G$.*

Remark: "Totally singular" submanifolds may have even larger, non-free symmetry groups, but these are not covered by the preceding results. See [25] for details and precise characterization of such submanifolds.

For example, the **Euclidean signature** for a curve in the Euclidean plane is the planar curve $\mathcal{S}(C) = \{(\kappa, \kappa_s)\}$ parametrized by the curvature invariant κ and its first derivative with respect to arc length. Two planar curves are equivalent under oriented rigid motions if and only if they have the same signature curves.

The maximally symmetric curves have constant Euclidean curvature, and so their signature curve degenerates to a single point. These are the circles and straight lines, and, in accordance with Theorem 13, each is the orbit of its one-parameter symmetry subgroup of SE(2). The number of Euclidean symmetries of a curve is equal to its index – the number of times the Euclidean signature is retraced as we go around the curve.

Thus, the signature curve method has the potential to be of practical use in the general problem of object recognition and symmetry classification. It offer several advantages over more traditional approaches. First, it is purely local, and therefore immediately applicable to occluded objects. Second, it provides a mechanism for recognizing symmetries and approximate symmetries of the object. See C. Shakiban's contribution to these proceedings for a discussion of applications of signature curves in DNA modeling. The design of a suitably robust "signature metric" for practical comparison of signatures is the subject of ongoing research.

Example 14. Let us next consider the equivalence and symmetry problems for binary forms. According to the general moving frame construction in Example 9, the signature curve $\mathcal{S} = \mathcal{S}(q)$ of a function (polynomial) $u = q(x)$ is parametrized by the covariants J^2 and K, as given in (14). The following solution to the equivalence problem for complex-valued binary forms, [1, 21, 24], is an immediate consequence of the general equivalence Theorem 10.

Theorem 15. *Two nondegenerate complex-valued forms $q(x)$ and $\bar{q}(x)$ are equivalent if and only if their signature curves are identical: $\mathcal{S}(q) = \mathcal{S}(\bar{q})$.*

All equivalence maps $\bar{x} = \varphi(x)$ solve the two rational equations

$$J(x)^2 = \bar{J}(\bar{x})^2, \qquad K(x) = \bar{K}(\bar{x}). \tag{15}$$

In particular, the theory guarantees φ is necessarily a linear fractional transformation!

Theorem 16. *A nondegenerate binary form $q(x)$ is maximally symmetric if and only if it satisfies the following equivalent conditions:*

a) *q is complex-equivalent to a monomial x^k, with $k \neq 0, n$*
b) *The covariant T^2 is a constant multiple of $H^3 \not\equiv 0$*
c) *The signature is just a single point*
d) *q admits a one-parameter symmetry group*
e) *The graph of q is the orbit of a one-parameter subgroup of* GL(2).

Binary forms that are not complex-equivalent to a monomial have only a finite symmetry group. In [1], Irina Kogan and I gave a practical method for computing the discrete symmetries of such forms by solving the rational equations (15). In her thesis [17], she further extended these results to forms in several variables. In particular, a complete signature for ternary forms leads to a practical algorithm for computing discrete symmetries of, among other cases, elliptic curves.

5. Joint Invariants and Differential Invariants

One practical difficulty with the differential invariant signature is its dependence upon high order derivatives, which makes it very sensitive to data noise. For this reason, a new signature paradigm, based on joint invariants, was proposed in [26]. We consider now the joint action

$$g \cdot (z_0, \ldots, z_n) = (g \cdot z_0, \ldots, g \cdot z_n), \qquad g \in G, \quad z_0, \ldots, z_n \in M \tag{16}$$

of the group G on the $(n+1)$-fold Cartesian product $M^{\times(n+1)} = M \times \cdots \times M$. An invariant $I(z_0, \ldots, z_n)$ of (16) is an $(n+1)$-*point joint invariant* of the original transformation group. In most cases of interest, although not in general, if G acts effectively on M, then, for $n \gg 0$ sufficiently large, the product action is free and regular on an open subset of $M^{\times(n+1)}$. Consequently, the moving frame method outlined in Section 2 can be applied to such joint actions, and thereby establish complete classifications of joint invariants and, via prolongation to Cartesian products of jet spaces, joint differential invariants. We will discuss two particular examples – planar curves in Euclidean geometry and projective geometry, referring to [26] for details.

Example 17. *Euclidean joint differential invariants.* Consider the proper Euclidean group SE(2) acting on oriented curves in the plane $M = \mathbb{R}^2$. We begin with the Cartesian product action on $M^{\times 2} \simeq \mathbb{R}^4$. Taking the simplest cross-section $x_0 = u_0 = x_1 = 0$, $u_1 > 0$ leads to the normalization equations

$$\begin{aligned} y_0 &= x_0 \cos\theta - u_0 \sin\theta + a = 0 \\ v_0 &= x_0 \sin\theta + u_0 \cos\theta + b = 0 \\ y_1 &= x_1 \cos\theta - u_1 \sin\theta + a = 0. \end{aligned} \tag{17}$$

Solving, we obtain a right moving frame

$$\theta = \tan^{-1}\left(\frac{x_1 - x_0}{u_1 - u_0}\right), \qquad a = -x_0 \cos\theta + u_0 \sin\theta, \qquad b = -x_0 \sin\theta - u_0 \cos\theta \tag{18}$$

along with the fundamental interpoint distance invariant

$$v_1 = x_1 \sin\theta + u_1 \cos\theta + b \quad \longmapsto \quad I = \|z_1 - z_0\|. \tag{19}$$

Substituting (18) into the prolongation formulae (7) leads to the the normalized first and second order joint differential invariants

$$\begin{aligned} \frac{dv_k}{dy} &\longmapsto J_k = -\frac{(z_1 - z_0)\cdot \dot z_k}{(z_1 - z_0)\wedge \dot z_k} \\ \frac{d^2 v_k}{dy^2} &\longmapsto K_k = -\frac{\|z_1 - z_0\|^3(\dot z_k \wedge \ddot z_k)}{[(z_1 - z_0)\wedge \dot z_0]^3} \end{aligned} \tag{20}$$

for $k = 0, 1$. Note that

$$J_0 = -\cot\phi_0, \qquad J_1 = +\cot\phi_1 \tag{21}$$

where $\phi_k = \sphericalangle(z_1 - z_0, \dot z_k)$ denotes the angle between the chord connecting z_0, z_1 and the tangent vector at z_k. The modified second order joint differential invariant

$$\widehat{K}_0 = -\|z_1 - z_0\|^{-3} K_0 = \frac{\dot z_0 \wedge \ddot z_0}{[(z_1 - z_0)\wedge \dot z_0]^3} \tag{22}$$

equals the ratio of the area of triangle whose sides are the first and second derivative vectors $\dot z_0$, $\ddot z_0$ at the point z_0 over the *cube* of the area of triangle whose sides are the chord from z_0 to z_1 and the tangent vector at z_0.

On the other hand, we can construct the joint differential invariants by invariant differentiation of the basic distance invariant (19). The normalized invariant differential operators are

$$D_{y_k} \longmapsto \mathcal{D}_k = -\frac{\|z_1 - z_0\|}{(z_1 - z_0)\wedge \dot z_k} D_{t_k}. \tag{23}$$

Proposition 18. *Every two-point Euclidean joint differential invariant is a function of the interpoint distance $I = \|z_1 - z_0\|$ and its invariant derivatives with respect to* (23).

A generic product curve $\mathbf{C} = C_0 \times C_1 \subset M^{\times 2}$ has joint differential invariant rank $2 = \dim \mathbf{C}$, and its joint signature $\mathcal{S}^{(2)}(\mathbf{C})$ will be a two-dimensional submanifold parametrized by the joint differential invariants I, J_0, J_1, K_0, K_1 of order ≤ 2. There will exist a (local) syzygy $\Phi(I, J_0, J_1) = 0$ among the three first order joint differential invariants.

Theorem 19. *A curve C or, more generally, a pair of curves $C_0, C_1 \subset \mathbb{R}^2$, is uniquely determined up to a Euclidean transformation by its reduced joint signature, which is parametrized by the first order joint differential invariants I, J_0, J_1. The curve(s) have a one-dimensional symmetry group if and only if their signature is a one-dimensional curve if and only if they are orbits of a common one-parameter subgroup $($i.e. concentric circles or parallel straight lines$)$; otherwise the signature is a two-dimensional surface, and the curve(s) have only discrete symmetries.*

For $n > 2$ points, we can use the two-point moving frame (18) to construct the additional joint invariants

$$y_k \longmapsto H_k = \|z_k - z_0\| \cos \psi_k, \qquad v_k \longmapsto I_k = \|z_k - z_0\| \sin \psi_k$$

where $\psi_k = \sphericalangle(z_k - z_0, z_1 - z_0)$. Therefore, a complete system of joint invariants for SE(2) consists of the angles ψ_k, $k \geq 2$, and distances $\|z_k - z_0\|$, $k \geq 1$. The other interpoint distances can all be recovered from these angles; vice versa, given the distances, and the sign of one angle, one can recover all other angles. In this manner, we establish the "First Main Theorem" for joint Euclidean differential invariants.

Theorem 20. *If $n \geq 2$, then every n-point joint* E(2) *differential invariant is a function of the interpoint distances $\|z_i - z_j\|$ and their invariant derivatives with respect to* (23). *For the proper Euclidean group* SE(2), *one must also include the sign of one of the angles, say $\psi_2 = \sphericalangle(z_2 - z_0, z_1 - z_0)$.*

Generic three-pointed Euclidean curves still require first order signature invariants, [26]. To create a Euclidean signature based entirely on joint invariants, we take four points z_0, z_1, z_2, z_3 on our curve $C \subset \mathbb{R}^2$. There are six different interpoint distance invariants

$$\begin{aligned} a &= \|z_1 - z_0\|, & b &= \|z_2 - z_0\|, & c &= \|z_3 - z_0\| \\ d &= \|z_2 - z_1\|, & e &= \|z_3 - z_1\|, & f &= \|z_3 - z_2\| \end{aligned} \tag{24}$$

which parametrize the joint signature $\widehat{\mathcal{S}} = \widehat{\mathcal{S}}(C)$ that uniquely characterizes the curve C up to Euclidean motion. This signature has the advantage of requiring no differentiation, and so is not sensitive to noisy image data. There are two local syzygies

$$\Phi_1(a,b,c,d,e,f) = 0, \qquad \Phi_2(a,b,c,d,e,f) = 0 \tag{25}$$

among the the six interpoint distances. One of these is the universal **Cayley-Menger syzygy**

$$\det \begin{vmatrix} 2a^2 & a^2 + b^2 - d^2 & a^2 + c^2 - e^2 \\ a^2 + b^2 - d^2 & 2b^2 & b^2 + c^2 - f^2 \\ a^2 + c^2 - e^2 & b^2 + c^2 - f^2 & 2c^2 \end{vmatrix} = 0 \tag{26}$$

which is valid for all possible configurations of the four points, and is a consequence of their coplanarity, cf. [2, 20]. The second syzygy in (25) is curve-dependent and serves to effectively characterize the joint invariant signature. Euclidean symmetries of the curve, both continuous and discrete, are characterized by this joint signature. For example, the number of discrete symmetries equals the signature index – the number of points in the original curve that map to a single, generic point in $\mathcal{S}$.

A wide variety of additional cases, including curves and surfaces in two and three-dimensional space under the Euclidean, equi-affine, affine and projective groups, are investigated in detail in [26]. Applications of these methods to the recognition and detection of symmetries in polygons can be found in [4].

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