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CLASSICAL AND QUANTUM COLLECTIVE DYNAMICS OF DEFORMABLE OBJECTS. SYMMETRY AND INTEGRABILITY PROBLEMS

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Abstract. Discussed is affine model of collective degrees of freedom of multi-particle systems or continuous media. The novelty of our approach is that it is not only kinematics, i.e. geometry of degrees of freedom, that is invariant under affine group, but rather we study affinely-invariant geodetic models of such affine systems. It is shown that the dynamics of bounded elastic vibrations may be encoded in such geodetic models in the very form of the kinetic energy expression. Some special solutions like the relative equilibria are discussed. We start with the general approach to group-theoretical degrees of freedom and then discuss peculiarities of the affine group and certain other groups underlying collective dynamics.

1. Introduction

It is well-known that one is faced with rather serious analytical difficulties when dealing with complex systems, in particular multi-particle ones. In general there is no hope for analytical solutions and any qualitative or approximate analysis. Fortunately, quite often degrees of freedom of such systems are in a sense hierarchically ordered in such a way that a relatively small part of them is approximately decoupled from the remaining ones and ruled by approximately autonomous dynamics. And this dynamics gives an account of the main features of the object, relevant for the considered phenomena. Such hierarchy of degrees of freedom usually appears due to some peculiarities of intermolecular forces and quite often it has to do with geometry of the physical space or some other spaces relevant for the problem. The leading parameters deciding about the main dynamical features of the system are usually referred to as collective modes. The rules of the collective dynamics are either (more or less qualitatively) derived from the micromodel or somehow guessed on the basis of certain natural symmetry demands. The very idea of collective modes is that they depend on all one-particle variables and the latter ones enter generalized collective coordinates on essentially equal footing, with the same "strength", i.e. in a non-local way. The very idea of various moment approaches, virial coefficients, etc., is based on the non-local character of relevant modes [4, 6, 10]. Very often these collective variables are ruled by some symmetry groups. This is a very fortunate circumstance, because due to the analytical character of Lie groups, there is a hope for some rigorous or at least approximate solutions based on certain functional series and special functions. This concerns both the classical and quantum levels. It was not accidental that the first mechanical system with a non-trivial topology successfully quantized in the early years of quantum mechanics was the rigid body, the configuration space of which may be identified with the group of rotations or Euclidean motions.

2. Hamiltonian Systems on Lie Groups

Let us begin with the general description of systems with group-theoretic degrees of freedom [1, 2, 14]. We do not consider the general case of Hamiltonian systems with homogeneous spaces as configuration manifolds, but just concentrate on the special case when the corresponding group acts freely, i.e. if every point has a trivial isotropy group. More general systems on homogeneous spaces may be obtained by an appropriate quotient procedure. In the introductory remarks below we concentrate on linear groups formalism, because practically all Lie groups used in physics may be faithfully realized by finite-dimensional matrices. The only exception are covering groups $\widetilde{GL}(n,\mathbb{R})$ and $\widetilde{SL}(n,\mathbb{R})$ of the real linear and unimodular groups $GL(n,\mathbb{R})$ and $SL(n,\mathbb{R})$, $n>1$ (double coverings if $n>2$, infinite $\mathbb{Z}$-coverings if $n=2$). By the way, this fact was a reason of many confusions and misunderstandings in attempts of generalizing usual spinors onto affine framework so as to obtain the half-objects ruled by the mentioned groups. The final outcome was that such objects must be either infinite-dimensional if linearly ruled by the mentioned groups or finite-dimensional but ruled by non-linear realizations of $\widetilde{GL}(n,\mathbb{R})$ and $\widetilde{SL}(n,\mathbb{R})$.

In any case the formal language of linear groups is at least graphically simple and the general group-theoretic content independent of linearity is easily readable. So, let us consider a mechanical system the configuration space of which is identified with some (linear) Lie group G. Motions are described as sufficiently regular curves $\mathbb{R}\ni t\mapsto g(t)\in\mathbb{R}$; the corresponding generalized velocities will be denoted by $\dot{g}(t)$. As usual, it is convenient to use Lie-algebraic objects, i.e. velocities regularly translated to the group neutral element,

$$\Omega := \dot{g}g^{-1}, \qquad \hat{\Omega} := g^{-1}\dot{g}, \tag{1}$$

obviously,

$$\Omega = g\hat{\Omega}g^{-1} = \mathrm{Ad}_g \hat{\Omega}. \tag{2}$$

If G is non-Abelian, then Ω and $\hat{\Omega}$ are non-holonomic velocities, i.e. they are not time derivatives of any generalized coordinates.

Left and right regular translations L_k and R_k,

$$L_k(g) := kg, \qquad R_k(g) := gk \tag{3}$$

affect the above objects according to the rule:

$$\begin{aligned} L_k &: \Omega \mapsto k\Omega k^{-1} = \mathrm{Ad}_k \Omega, \qquad \hat{\Omega} \mapsto \hat{\Omega} \\ R_k &: \Omega \mapsto \Omega, \qquad \hat{\Omega} \mapsto k^{-1}\hat{\Omega}k = \mathrm{Ad}_k^{-1} \hat{\Omega} \end{aligned} \tag{4}$$

(adjoint transformation and invariance).

Obviously, Ω and $\hat{\Omega}$ are elements of the Lie algebra of G, $\Omega \in \mathfrak{g}$ and $\hat{\Omega} \in \mathfrak{g}$. They have to do respectively with the right- and left-invariant vector fields X and Y on G:

$$X_g[\Omega] := \Omega g, \qquad Y_g[\hat{\Omega}] := g\hat{\Omega}. \tag{5}$$

In the above formulas Ω and $\hat{\Omega}$ are fixed elements of $\mathfrak{g}$ and label the corresponding vector fields.

In canonical formalism one uses also the dual objects $\Sigma \in \mathfrak{g}^*$ and $\hat{\Sigma} \in \mathfrak{g}^*$. They are related to canonical momenta p and configurations g by the formulas

$$\langle \Sigma, \Omega \rangle = \langle \hat{\Sigma}, \hat{\Omega} \rangle = \langle p, \dot{g} \rangle \tag{6}$$

where obviously the bracket symbol denotes evaluation of covectors on vectors, $\dot{g} \in T_g G$, $p \in T_g^* G$ and g, $\dot{g}$ are arbitrary. The above formula implies that

$$\Sigma = \mathrm{Ad}_g^{*-1} \hat{\Sigma} \tag{7}$$

where Ad_g^* is the adjoint of Ad_g.

The objects Σ and $\hat{\Sigma}$ are respectively Hamiltonian generators of the groups of left and right regular translations L_G and R_G.

If, as assumed for simplicity, G is a linear group, $G \subset GL(W) \subset L(W)$ for some linear space (e.g. $\mathbb{R}^n$ or $\mathbb{C}^n$), then usually some technical simplifications occur. Namely, $L(W)^* \simeq L(W)$ in the sense of pairing

$$\langle C, D \rangle = \mathrm{Tr}(CD). \tag{8}$$

As $\mathfrak{g} \subset L(W)$, its dual has the form:

$$\mathfrak{g}^* \simeq L(W)^* / \operatorname{An} \mathfrak{g} \tag{9}$$

where, obviously, $\mathrm{An}\,\mathfrak{g}$ consists of functionals vanishing on $\mathfrak{g}$. According to the above trace formula, $\mathrm{An}\,\mathfrak{g}$ may be identified with some linear subspace $\mathfrak{g}^{\perp}$ of $L(W)$. Thus,

$$\mathfrak{g}^* \simeq L(W)/\mathfrak{g}^{\perp}. \tag{10}$$

And as a rule, in typical situations, this quotient space may be canonically identified with some distinguished linear subspace of $L(W)$ consisting of natural representants of cosets. It is very often so that $\mathfrak{g}^*$ may be identified with $\mathfrak{g}$ itself.

Transformation rules for Σ and $\hat{\Sigma}$ are analogous to those for Ω and $\hat{\Omega}$,

$$\begin{aligned} L_k &: \Sigma \mapsto \mathrm{Ad}^{*-1}_k \Sigma, & \hat{\Sigma} &\mapsto \hat{\Sigma} \\ R_k &: \Sigma \mapsto \Sigma, & \hat{\Sigma} &\mapsto \mathrm{Ad}^*_k \hat{\Sigma}. \end{aligned} \tag{11}$$

If the aforementioned identifications work, we have simply:

$$\begin{aligned} L_k &: \Sigma \mapsto k\Sigma k^{-1}, & \hat{\Sigma} &\mapsto \hat{\Sigma} \\ R_k &: \Sigma \mapsto \Sigma, & \hat{\Sigma} &\mapsto k^{-1}\hat{\Sigma}k. \end{aligned} \tag{12}$$

Σ and $\hat{\Sigma}$ are momentum mappings corresponding respectively to the groups of left and right regular translations. Obviously,

$$\Sigma = g\hat{\Sigma}g^{-1}. \tag{13}$$

Just as previously, Σ and $\hat{\Sigma}$ have to do with right- and left-invariant covector fields (differential one-forms) A and B on G. If the aforementioned identification works, then

$$A_g[\Sigma] = g^{-1}\Sigma, \qquad B_g[\hat{\Sigma}] = \hat{\Sigma}g^{-1}. \tag{14}$$

Poisson brackets of Σ are expressed through the structure constants of G, those of $\hat{\Sigma}$ have a reversed sign, and the mutual Poisson brackets of Σ- and $\hat{\Sigma}$-components vanish (because left and right regular translations mutually commute).

Kinetic energy T of a system with G-degrees of freedom is equivalent to some Riemann structure on G:

$$T = \frac{1}{2}\Gamma_{\mu\nu}(q)\dot{q}^{\mu}\dot{q}^{\nu} \tag{15}$$

where q^{μ} are generalized coordinates on G. In general the dynamical metric tensor Γ depends both on intrinsic geometry of G and on some physically motivated inertial parameters.

The theory of geodetic Hamiltonian systems on Lie groups developed by Hermann, Arnold and others deals with kinetic energies T (metrics Γ) invariant under left or right (or both) regular translations on G.

It is easy to see that any left-invariant T is a quadratic form of $\hat{\Omega}$ with constant coefficients. Similarly, right-invariant kinetic energies T are quadratic forms of Ω with constant coefficients. If G is non-Abelian, then the corresponding metric tensors Γ on G have non-vanishing curvature tensors. Of particular interest are highly-symmetric models, when T and Γ are simultaneously invariant under left and right regular translations.

When deriving and analyzing equations of motion it is more convenient to use Hamiltonian formalism and Poisson brackets then to base on Lagrange equations. For geodetic models, when Lagrangian L coincides with T, and for potential models, when $L = T - V(q)$, Legendre transformation has the form

$$p_\mu = \frac{\partial L}{\partial \dot{q}^\mu} = \Gamma_{\mu\nu}\dot{q}^\nu \tag{16}$$

and the corresponding Hamiltonian is given by:

$$H = \Im + V(q) = \frac{1}{2}\Gamma^{\mu\nu}p_\mu p_\nu + V(q) \tag{17}$$

where, obviously,

$$\Gamma^{\mu\lambda}\Gamma_{\lambda\nu} = \delta^\mu_\nu. \tag{18}$$

Calculations and analysis become simpler when expressing everything through non-holonomic velocities and momenta. As mentioned, for left-invariant kinetic energies we have:

$$T = \frac{1}{2}\pounds_{\mu\nu}\hat{\Omega}^\mu\hat{\Omega}^\nu \tag{19}$$

where $\pounds_{\mu\nu} = \pounds_{\nu\mu}$ are constant and $\hat{\Omega}^\mu$ are expansion coefficients of $\hat{\Omega}$ with respect to some basis $\{E_\mu\}$ in $\mathfrak{g}$. Similarly, for right-invariant models we have:

$$T = \frac{1}{2}\Re_{\mu\nu}\Omega^\mu\Omega^\nu \tag{20}$$

again $\Re_{\mu\nu} = \Re_{\nu\mu}$ being constants. In usual mechanical applications the matrices $\pounds$ and $\Re$ are symmetric and positively definite, although some hyperbolic-signature models may be also geometrically and physically interesting [1].

Expressing Legendre transformations in non-holonomic terms,

$$\Sigma_\mu = \frac{\partial T}{\partial \Omega^\mu} = \pounds_{\mu\nu}\Omega^\nu, \qquad \hat{\Sigma}_\mu = \frac{\partial T}{\partial \hat{\Omega}^\mu} = \Re_{\mu\nu}\hat{\Omega}^\nu \tag{21}$$

one obtains geodetic Hamiltonians in the form:

$$\Im = \frac{1}{2}\pounds^{\mu\nu}\hat{\Sigma}_\mu\hat{\Sigma}_\nu, \qquad \Im = \frac{1}{2}\Re^{\mu\nu}\Sigma_\mu\Sigma_\nu \tag{22}$$

where, obviously,

$$\pounds^{\mu\lambda}\pounds_{\lambda\nu} = \delta^\mu_\nu, \qquad \Re^{\mu\lambda}\Re_{\lambda\nu} = \delta^\mu_\nu. \tag{23}$$

If potential is admitted, then, obviously, the total Hamiltonian is given by: $H = \Im + V(q)$.

As mentioned, we have the following Poisson brackets:

$$\{\Sigma_\mu, \Sigma_\nu\} = C^\lambda_{\mu\nu}\Sigma_\lambda, \qquad \{\hat{\Sigma}_\mu, \hat{\Sigma}_\nu\} = -C^\lambda_{\mu\nu}\hat{\Sigma}_\lambda, \qquad \{\Sigma_\mu, \hat{\Sigma}_\nu\} = 0. \tag{24}$$

For any function F depending only on coordinates q, we have:

$$\{\Sigma_\mu, F\} = -L_\mu F, \qquad \{\hat{\Sigma}_\mu, F\} = -R_\mu F \tag{25}$$

where L_μ and R_μ are differential operators generating left and right regular translations on G. Thus, if q^μ are canonical coordinates of the first kind on G ($g(q) = \mathrm{e}^{q^\mu E_\mu}$), then:

$$\frac{\partial}{\partial q^\mu} F(k(q)g)|_{q=0} = (L_\mu F)(g), \qquad \frac{\partial}{\partial q^\mu} F(gk(q))|_{q=0} = (R_\mu F)(g) \tag{26}$$

and

$$[L_\mu, L_\nu] = C^\lambda_{\mu\nu} L_\lambda, \qquad [R_\mu, R_\nu] = -C^\lambda_{\mu\nu} R_\lambda, \qquad [L_\mu, R_\nu] = 0. \tag{27}$$

Let us quote a few examples of group-theoretic configuration spaces of collective modes and internal degrees of freedom.

1. $G = SO(n, \mathbb{R})$ – n-dimensional rigid body without translational motion. Ω and $\hat{\Omega}$ are skew-symmetric. Their matrix elements are, respectively, components of the angular velocity with respect to the space-fixed and body-fixed reference frames. Obviously, only the special cases $n = 2$ and $n = 3$ are directly interpretable in physical terms. Actually $\mathfrak{so}(n)^*$ may be canonically identified with $\mathfrak{so}(n)$ itself through the trace formula, or better, in the convention $\langle A, B\rangle = \frac{1}{2}\operatorname{Tr}(AB)$. The skew-symmetric dual objects Σ and $\hat{\Sigma}$ describe the rotational angular momentum respectively in terms of the space-fixed and body-fixed reference frames. Left regular translations describe the rigid body spatial rotations, whereas the right translations permute material points without affecting the body orientation in space. The usual formula for the kinetic energy reads

$$T = -\frac{1}{2}\operatorname{Tr}(\hat{\Omega}^2 J) \tag{28}$$

where J is a constant symmetric positively definite matrix describing the rotational inertia. It is obtained as the second order moment of the mass distribution with respect to the co-moving frame. The corresponding T is left-invariant, i.e. non-sensitive with respect to spacial rotations. It becomes also right-invariant when the top is spherical, i.e. J is proportional to the identity matrix. In general, degeneracy of J corresponds to the invariance with respect to the right action of certain subgroups of $SO(n, \mathbb{R})$. If $n = 3$ and J is once degenerate, we are dealing with the symmetric top.
The corresponding geodetic Hamiltonians are given by

$$\Im = -\frac{1}{2}\operatorname{Tr}(\hat{\Sigma}^2 J^{-1}). \tag{29}$$

2. $G = E(n, \mathbb{R}) = SO(n, \mathbb{R}) \times_s \mathbb{R}^n$ – n-dimensional rigid body with translational motion.
3. Affinely-rigid (homogeneously-deformable) body:
$$G = G\,\mathrm{Aff}(n, \mathbb{R}) = GL(n, \mathbb{R}) \times_s \mathbb{R}^n.$$
Without translational motion: $G = GL(n, \mathbb{R})$, and then $\mathfrak{g} = L(n, \mathbb{R})$ and $\mathfrak{g}^* \simeq L(n, \mathbb{R})$.
4. Incompressible affinely-rigid body: $G = SL(n, \mathbb{R})$ – without translations, $\mathfrak{g} = \mathfrak{sl}(n) \simeq \mathfrak{sl}(n)^*$ – traceless matrices. With translational motion: $G = SL(n, \mathbb{R}) \times_s \mathbb{R}^n$.
5. Projectively-rigid body: $G = Pr(n, \mathbb{R}) \simeq SL(n + 1, \mathbb{R})$.
6. Complex matrices, less typical in mechanics: $G = U(n)$ – unitary group; unitary-rigid body $\mathfrak{g} = \mathfrak{u}(n) \simeq \mathfrak{u}(n)^*$ – antihermitian matrices, i.e. ones satisfying: $A^\dagger = -A$.
$G = GL(n, \mathbb{C})$ – complexified affinely-rigid body. $\mathfrak{g} \simeq L(n, \mathbb{C}) \simeq L(n, \mathbb{C})^*$.
The real Lie groups $GL(n, \mathbb{R})$ and $U(n)$ are two different (in a sense opposite) real forms of the same complex Lie group $GL(n, \mathbb{C})$. Both were suggested by Westpfahl as models of internal and collective degrees of freedom.
7. Ideal incompressible fluid. This is an infinite-dimensional system. The corresponding group $G = \mathrm{SDiff}(\mathbb{R}^n)$ consists of all volume-preserving diffeomorphisms of $\mathbb{R}^n$ onto itself ($n = 3$ in the physical case). This description of ideal fluids is due to Arnold. It was very helpful at least as a heuristic tool for searching solutions, although the theory of infinite-dimensional Lie groups is still far from being complete. Roughly speaking, $\mathfrak{g}$, Lie algebra of G consists of vector fields with the vanishing divergence. The geodetic Hamiltonian system on $\mathrm{SDiff}(\mathbb{R}^n)$ underlying the ideal fluid dynamics is right-invariant [2]. For a compressible continuous medium we would have to use $G = \mathrm{Diff}(\mathbb{R}^n)$ as a configuration space.

The main objective of our study is an analysis of the above items 3, 4, 5 and partially 6.

The Poisson brackets quoted above provide a convenient tool for deriving equations of motion in the Hamilton-Poisson form:
$$\frac{\mathrm{d}F}{\mathrm{d}t} = \{F, H\} \tag{30}$$
where H is a Hamiltonian, e.g. in a potential form, $H = \Im + V(q)$, and F runs over some systems of coordinates in the phase space manifold, i.e. some maximal system of functionally independent functions.

The usual Schrödinger quantization of geodetic systems on Riemannian manifolds (Q, Γ) is based on the use of $L^2(G, \Gamma)$ as the Hilbert space of wave functions. Obviously, it consists of complex-valued functions on Q which are square-integrable in the sense of natural measure μ_Γ induced on Q by Γ,

$$\mathrm{d}\mu_\Gamma(q) = \sqrt{|\det[\Gamma_{\mu\nu}]|}\, \mathrm{d}q^1 \ldots \mathrm{d}q^f, \qquad f = \dim Q. \tag{31}$$

Scalar product is defined by the usual formula

$$\langle \Psi_1 \mid \Psi_2 \rangle = \int \overline{\Psi}_1(q) \Psi_2(q)\, \mathrm{d}\mu_\Gamma(q). \tag{32}$$

If the classical kinetic energy is given by (15), then the corresponding quantum operator $\mathbf{T}$ is usually postulated in the form

$$\mathbf{T} = -\frac{\hbar^2}{2} \Delta[\Gamma] \tag{33}$$

where $\Delta[\Gamma]$ is the Laplace-Beltrami operator induced by Γ

$$\Delta[\Gamma] = \Gamma^{\mu\nu} \nabla_\mu \nabla_\nu = \frac{1}{\sqrt{|\Gamma|}} \sum_{\mu\nu} \partial_\mu \sqrt{|\Gamma|} \Gamma^{\mu\nu} \partial_\nu. \tag{34}$$

Obviously, in this formula $|\Gamma|$ is an abbreviation for $|\det[\Gamma_{\mu\nu}]|$ and ∇_μ denotes the Levi-Civita covariant differentiation in the sense of Γ. Roughly speaking, (33) is based on the heuristic replacement

$$p_\mu \mapsto \mathbf{p}_\mu = \frac{\hbar}{\mathrm{i}} \nabla_\mu. \tag{35}$$

For the classical potential system with the Lagrangian $L = T - V(q)$ the corresponding quantum Hamiltonian is given by

$$\mathbf{H} = \mathbf{T} + V(q). \tag{36}$$

There are also certain geometric and quasi-classical arguments for introducing an additional potential term proportional to the scalar curvature of Γ. However, even if justified, it is non-essential in constant-curvature spaces, which are particularly interesting for us, because results then merely in an overall shift of energy levels.

This is the general pattern for Riemannian configuration spaces. If the manifolds Q is a Lie group G with some left- or right- (or both) invariant kinetic energy like (19) or (20), i.e. in the canonical language (22), then it is convenient to use operators $\mathbf{\Sigma}_\mu$ and $\hat{\mathbf{\Sigma}}_\mu$ given by:

$$\mathbf{\Sigma}_\mu = \frac{\hbar}{\mathrm{i}} L_\mu, \qquad \hat{\mathbf{\Sigma}}_\mu = \frac{\hbar}{\mathrm{i}} R_\mu. \tag{37}$$

They are generators of the corresponding transformation groups acting on the wave function arguments. Quantum kinetic energy is then given by

$$\mathbf{T} = \frac{1}{2}\pounds_{\mu\nu}\mathbf{\Sigma}_\mu\mathbf{\Sigma}_\nu, \qquad \mathbf{T} = \frac{1}{2}\Re_{\mu\nu}\hat{\mathbf{\Sigma}}_\mu\hat{\mathbf{\Sigma}}_\nu \tag{38}$$

respectively for two versions of (25).

3. Geometric Description of Affinely-Rigid Body

Strictly speaking, configuration spaces of mechanical systems quoted above are not Lie groups but their homogeneous spaces, in those cases with trivial isotropy groups. They may be identified with Lie groups when some standard configuration is fixed. This is just like the difference between affine and linear spaces. This mathematical purity might seem superfluous on the level of computation, nevertheless, some conceptual and even just analytical mistakes are possible when one neglects the mentioned distinction. Below we present the correct description of affinely-rigid body in geometric terms. We shall follow the language of continua, although in principle discrete systems may be also described in these terms [3, 5, 9, 10, 11, 12, 13, 14, 15, 16].

We are given two Euclidean spaces (N, U, η) and (M, V, g), the material and physical space, respectively. Here N and M are the basic point spaces, U and V are their linear translation spaces, and $\eta \in U^* \otimes U^*$, $g \in V^* \otimes V^*$ are their metric tensors. Here N is used for labelling the material points, and elements of M are geometric spatial points.

The configuration space of affinely-rigid body

$$Q := \mathrm{Aff}\, I(N, M) \tag{39}$$

consists of affine isomorphisms of N onto M. The material labels $a \in N$ are parameterized by Cartesian coordinates a^K-Lagrange variables. Cartesian coordinates in M will be denoted by y^i, and the corresponding geometric points by y. The configuration $\Phi \in Q$ is to be understood in such a way that the material point $a \in N$ occupies the spatial position $y = \Phi(a)$.

Let μ denote the co-moving (Lagrangian) mass distribution in N; obviously, it is constant in time. Lagrangian coordinates a^K in N will be always chosen in such a way that their origin $a^K = 0$ coincides with the centre of mass $\mathcal{C}$

$$\int a^K \,\mathrm{d}\mu(a) = 0. \tag{40}$$

The configuration space may be identified then with $M \times LI(U, V)$,

$$Q = \mathrm{Aff}\, I(N, M) \simeq M \times LI(U, V) = M \times Q_{\mathrm{int}} \tag{41}$$

where $LI(U, V)$ denotes the manifold of all linear isomorphisms of U onto V. The Cartesian product factors refer respectively to the translational motion (M) and the internal or relative motion $(LI(U, V))$. We can also use another convenient representation, namely,

$$Q \simeq M \times F(V) = M \times Q_{\text{int}} \tag{42}$$

where $F(V)$ denotes the manifold of all linear frames in V. This is equivalent to putting $U = \mathbb{R}^n$ (vectors in V are identifiable with linear mappings from $\mathbb{R}$ to V; linear frames in V may be identified with isomorphisms of $\mathbb{R}^n$ onto V). Such a representation is unavoidable when extended bodies are replaced by point-like objects with extra attached internal degrees of freedom. Motion is described as a continuum of instantaneous configurations

$$\Phi(t, a)^i = \varphi^i_K(t) a^K + x^i(t) \tag{43}$$

where $x(t)$ is the centre of mass position, and $\varphi(t)$ tells us how constituents of the body are placed with respect to the centre of mass. The quantities (x^i, φ^i_K) are our generalized coordinates q^μ. Generalized velocities $(\dot{q}^\mu) = (\dot{x}^i, \dot{\varphi}^i_K)$ will be shortly denoted by $(\dot{x}, \dot{\varphi})$ or (v, ξ).

Obviously, if we put $U = V = \mathbb{R}^n$, then Q reduces to the aforementioned $G\,\mathrm{Aff}(n, \mathbb{R}) \simeq GL(n, \mathbb{R}) \times_s \mathbb{R}^n$, and Q_{int} reduces to $GL(n, \mathbb{R})$.

Inertia of affinely-constrained systems of material points is described by two constant quantities

$$m = \int \mathrm{d}\mu(a), \qquad J^{KL} = \int a^K a^L \,\mathrm{d}\mu(a) \tag{44}$$

i.e. the total mass m and the co-moving second-order moment $J \in U \otimes U$. More precisely, it is so in the usual theory based on the d'Alembert principle, when the kinetic energy is obtained by summation (integration) of usual (based on the metric g) kinetic energies of constituents [9, 10, 11, 12, 13, 14, 15, 16],

$$T = \frac{1}{2} g_{ij} \int \frac{\partial \Phi^i}{\partial t} \frac{\partial \Phi^j}{\partial t} \,\mathrm{d}\mu(a). \tag{45}$$

Substituting to this general formula the above affine constraints (43) we obtain

$$T = T_{\text{tr}} + T_{\text{int}} = \frac{m}{2} g_{ij} \frac{\mathrm{d}x^i}{\mathrm{d}t} \frac{\mathrm{d}x^j}{\mathrm{d}t} + \frac{1}{2} g_{ij} \frac{\mathrm{d}\varphi^i_A}{\mathrm{d}t} \frac{\mathrm{d}\varphi^i_B}{\mathrm{d}t} J^{AB}. \tag{46}$$

Obviously, if we analytically identify U and V with $\mathbb{R}^n$, and $LI(U, V)$ with $GL(n, \mathbb{R})$, then

$$T_{\text{int}} = \frac{1}{2} \mathrm{Tr}(\dot{\varphi}^T \dot{\varphi} J). \tag{47}$$

When we impose metrical rigidity constraints, $\varphi \in SO(n, \mathbb{R})$, then $\hat{\Omega}$ becomes skew-symmetric (just as Ω), $\hat{\Omega}^T = -\hat{\Omega}$, $\Omega^T = -\Omega$, and the formula for T_{int} reduces to (28).

The phase space of affinely-rigid body may be identified with

$$P := M \times LI(U, V) \times V^* \times L(V, U) \tag{48}$$

where the pairing between canonical momenta $(p, \pi) \in V^* \times L(V, U)$ and velocities $(v, \xi) \in V \times L(U, V)$ is understood, obviously, as:

$$\langle (p, \pi), (v, \xi) \rangle := \langle p, v \rangle + \mathrm{Tr}(\pi\xi) = p_i v^i + \pi_i{}^A \xi^i{}_A. \tag{49}$$

For Lagrangians of the form $L = T - \mathcal{V}(x, \varphi)$ (velocity-independent potentials) Legendre transformation has the form

$$p_i = m g_{ij} v^j, \qquad \pi_i^A = g_{ij} \xi^j_B J^{BA}. \tag{50}$$

The corresponding Hamiltonians have the form

$$H = \Im + \mathcal{V} = \frac{1}{2m} g^{ij} p_i p_j + \frac{1}{2} \tilde{J}_{AB} \pi_i^A \pi_j^B + \mathcal{V} \tag{51}$$

where $\tilde{J}_{AC} J^{CB} = \delta^B_A$.

Within the framework of this geometric description, the left and right regular translations (3) are replaced by superpositions of $\Phi \in Q = \mathrm{Aff}\, I(N, M)$ respectively with elements of affine groups $G\,\mathrm{Aff}(M)$ (spatial transformations) and $G\,\mathrm{Aff}(N)$ (material transformations). In particular, the corresponding transformations of the internal configuration space have the form:

$$A \in GL(V) : \varphi \mapsto A\varphi, \qquad B \in GL(U) : \varphi \mapsto \varphi B \tag{52}$$

for any $\varphi \in Q_{\mathrm{int}} = LI(U, V)$. Obviously, this reduces to (3) when we put $U = V = \mathbb{R}^n$ and $LI(U, V) = GL(n, \mathbb{R})$.

Putting $\mathcal{V} = 0$ we obtain some geodetic systems on $Q = \mathrm{Aff}\, I(N, M)$ or $Q_{\mathrm{int}} = LI(U, V)$. These systems (metrics on Q, Q_{int}) are never invariant either under left or right translations. Therefore, they are not invariant group-theoretical systems in the Arnold-Hermann sense. They are not interesting in geometric sense, because the very taste and beauty of systems with the group-theoretic background consists just in their invariance properties. And besides, geodetic systems based on the above T are non-physical, because they predict the unlimited contraction or expansion. In particular, the singular configurations with $\det \varphi = 0$ are admissible. Such purely geodetic models, without the extra imposed potentials $\mathcal{V}$ cannot describe anything like elastic vibrations or any kind of bounded motion.

Nevertheless, Hamiltonians with the above kinetic energy form (derived from the d'Alembert principle and formal substitution of constraints to the primary multiparticle expression) are physically applicable when an appropriately chosen potential energy term is included. Also dissipative phenomena with internal and external friction may be described. There is a wide spectrum of applications in various branches of physics and various scales of physical phenomena:

- astrophysics: figures of equilibrium, shape of Earth, vibrations of stars and gas clouds of interstellar matter,
- hydrodynamics: vibrating transversal cross-sections of fluid streams, fluid-droplet suspensions in other fluids (when no-mixing),
- macroscopic elasticity: situations when the length of excited waves is comparable with the linear size of the body, micromorphic continua with internal degrees of freedom,
- molecular dynamics: vibrations of molecules and their interaction with rotations,
- nuclear dynamics: collective droplet models of nuclei.

Non-holonomic velocities introduced previously for systems on Lie groups have now the following status

$$\Omega = \frac{\mathrm{d}\varphi}{\mathrm{d}t}\varphi^{-1} \in L(V), \qquad \hat{\Omega} = \varphi^{-1}\frac{\mathrm{d}\varphi}{\mathrm{d}t} \in L(U). \tag{53}$$

We shall use the term "affine velocity", respectively in spatial (current) and co-moving (material) representation. When the motion is rigid, i.e. constrained by the condition:

$$\eta = \varphi^* g, \qquad \text{i.e.} \qquad \eta_{AB} = g_{ij}\varphi^i{}_A\varphi^j{}_B \tag{54}$$

then Ω and $\hat{\Omega}$ become respectively g-skew-symmetric and η-skew-symmetric,

$$\Omega^i{}_j = -g_{jk}\Omega^k{}_l g^{li}, \qquad \hat{\Omega}^A{}_B = -\eta_{BC}\hat{\Omega}^C{}_D\eta^{DA} \tag{55}$$

i.e. reduce to angular velocity representations with respect to the space-fixed and body-fixed frames.

From the point of view of continuum mechanics, Ω represents the Euler velocity field. This means that the velocity of a particle instantaneously placed at $y \in M$ is given by

$$^{E}v(y)^i = \frac{\mathrm{d}x^i}{\mathrm{d}t} + \Omega^i{}_j(y^j - x^j). \tag{56}$$

In certain formulas it is convenient to use also the co-moving representation of translational velocity

$$\hat{v}^A := \varphi^{-1A}{}_i v^i = \varphi^{-1A}{}_i \frac{\mathrm{d}x^i}{\mathrm{d}t}. \tag{57}$$

Non-holonomic momenta conjugate to Ω and $\hat{\Omega}$ have the same geometric status: $\Sigma \in L(V)$, $\hat{\Sigma} \in L(U)$ and the pairing is given by

$$\langle \Sigma, \Omega \rangle = \langle \hat{\Sigma}, \hat{\Omega} \rangle = \mathrm{Tr}(\Sigma\Omega) = \mathrm{Tr}(\hat{\Sigma}\hat{\Omega}). \tag{58}$$

This means that the Lie algebras $\mathfrak{gl}(V) \simeq L(V)$, $\mathfrak{gl}(U) \simeq L(U)$ and their conjugate spaces $L(V)^*$, $L(U)^*$ are pairwise identified. Analytically

$$\Sigma^i{}_j = \varphi^i{}_A \pi^A{}_j, \qquad \hat{\Sigma}^A{}_B = \pi^A{}_i \varphi^i{}_B. \tag{59}$$

Obviously,

$$\langle \Sigma, \Omega \rangle = \langle \hat{\Sigma}, \hat{\Omega} \rangle = \langle \pi, \xi \rangle. \tag{60}$$

The quantities Σ and $\hat{\Sigma}$ are referred to as canonical affine spin respectively in the space-fixed and body-fixed reference frames. They are Hamiltonian generators of spatial and material transformations (52). The quantities Ω, $\hat{\Omega}$, Σ, $\hat{\Sigma}$ are affected by these transformations exactly as in formulas (4), (12) or more precisely

$$\begin{aligned} A \in GL(V): \quad & \Omega \mapsto A\Omega A^{-1}, \quad \Sigma \mapsto A\Sigma A^{-1}, \quad \hat{\Omega} \mapsto \hat{\Omega}, \quad \hat{\Sigma} \mapsto \hat{\Sigma} \\ B \in GL(U): \quad & \Omega \mapsto \Omega, \quad \Sigma \mapsto \Sigma, \quad \hat{\Omega} \mapsto B^{-1}\hat{\Omega}B, \quad \hat{\Sigma} \mapsto B^{-1}\hat{\Sigma}B. \end{aligned} \tag{61}$$

Just as in the formalism of Lie groups as configuration spaces, Σ and $\hat{\Sigma}$ are momentum mappings of $GL(V)$ and $GL(U)$ as transformation groups. The orthogonal subgroups (rotations) $SO(V,g)$ and $SO(U,\eta)$ are generated by the canonical spin S and canonical vorticity V, respectively,

$$S^i{}_j = \Sigma^i{}_j - g^{ik} g_{jm} \Sigma^m{}_k, \qquad V^A{}_B = \hat{\Sigma}^A{}_B - \eta^{AC} \eta_{BD} \hat{\Sigma}^D{}_C. \tag{62}$$

Remark: In general $S^i{}_j \neq \varphi^i{}_A V^A{}_B \varphi^{-1B}{}_j$, although $\Sigma^i{}_j = \varphi^i{}_A \hat{\Sigma}^A{}_B \varphi^{-1B}{}_j$. The equality holds only for the rigid motion, when $\varphi \in SO(n,\mathbb{R})$. And then $V^A{}_B$ become identical with $\hat{S}^A{}_B$, i.e. the co-moving components of canonical spin.

Obviously, S is the doubled g-skew-symmetric part of Σ, and V is the doubled η-skew-symmetric part of $\hat{\Sigma}$. The symmetric parts of S and V (in the sense of g and η, respectively) generate deformative transformations.

Various deformation measures are used in continuum mechanics. Let us quote their form in the special case of a homogeneously deformable (affinely-rigid) body. Green deformation tensor $G \in U^* \otimes U^*$ and Cauchy deformation tensor $C \in V^* \otimes V^*$ are defined as $G = \varphi^* g$ and $C = \varphi^{-1*}\eta$, i.e.

$$G_{AB} = g_{ij} \varphi^i{}_A \varphi^j{}_B, \qquad C_{ij} = \eta_{AB} \varphi^{-1A}{}_i \varphi^{-1B}{}_j. \tag{63}$$

In certain formulas their inverses $\tilde{G} \in U \otimes U$ and $\tilde{C} \in V \otimes V$ are used,

$$\tilde{G}^{AC} G_{CB} = \delta^A{}_B, \qquad \tilde{C}^{ik} C_{kj} = \delta^i{}_j. \tag{64}$$

Remark: $\tilde{G}^{AB} \neq G_{CD} \eta^{CA} \eta^{DB}$, $\tilde{C}^{ij} \neq C_{kl} g^{ki} g^{lj}$, because of this one must be careful with the use of upper-case and lower-case convention.

The Lagrange and Euler deformation tensors $E \in U^* \otimes U^*$ and $e \in V^* \otimes V^*$ are defined as follows:

$$E := \frac{1}{2}(G - \eta), \qquad e := \frac{1}{2}(g - C). \tag{65}$$

They vanish when there is no deformation, i.e. $\varphi \in LI(U,V)$. Green and Cauchy tensors reduce then to the corresponding metrics, $G = \eta$ and $C = g$. It is important that the definition of G is independent of η, whereas that of C does not depend on

g. Therefore, the really convincing deformation measures are those given by E and e, because they establish the relationship between preestablished metrics η, g and the φ-transferred metrics $\varphi^* g$, $\varphi^{-1*}\eta$, respectively. Without metrics η, g we would deal with a system with affine degrees of freedom, but neither rotational nor deformative content of φ could be defined. Therefore, from the point of view of geometry of degrees of freedom, the term "affinely-rigid" is more fundamental than "homogeneously-deformable".

We often need some scalar measures of deformation, i.e. coordinate-independent quantities built of deformation tensors. They tell us how strongly body is deformed, but do not contain any information as to how the deformation is oriented in space (M) and in the body (N). In n dimensions there are n functionally independent basic invariants. They may be chosen in a variety of ways, e.g.

$$\mathrm{Tr}(\hat{G}^k), \qquad \mathrm{Tr}(\hat{C}^k), \qquad \mathrm{Tr}(\hat{E}^k), \qquad \mathrm{Tr}(\hat{e}^k), \qquad k = \overline{1,n} \tag{66}$$

where $\hat{G} \in U \otimes U^*$, $\hat{C} \in V \otimes V^*$, $\hat{E} \in U \otimes U^*$, $\hat{e} \in V \otimes V^*$, and

$$\hat{G}^A_B := \eta^{AC} G_{CD}, \quad \hat{C}^i_j := g^{ik} C_{kj}, \quad \hat{E}^A_B := \eta^{AC} E_{CD}, \quad \hat{e}^i_j := g^{ik} e_{kj}. \tag{67}$$

Other possible and popular choices are solutions of the eigenequations:

$$\begin{aligned} \det[\hat{G}^A_B - \lambda\delta^A_B] &= 0, \qquad \det[\hat{C}^i_j - \lambda\delta^i_j] = 0 \\ \det[\hat{E}^A_B - \lambda\delta^A_B] &= 0, \qquad \det[\hat{e}^i_j - \lambda\delta^i_j] = 0. \end{aligned} \tag{68}$$

Left-hand sides of these equations are n-th order polynomials of λ. Their coefficients at λ^p, $p = 0, 1, \ldots, (n-1)$, provide another convenient choice of deformation invariants. Spatial and material isometries, $\varphi \mapsto A\varphi$, $\varphi \mapsto \varphi B$, where $A \in SO(V, g)$, $B \in SO(U, \eta)$ do not affect deformation invariants; this is just the reason they were called so.

In certain problems it is convenient to use the translational and total affine momentum with respect to some fixed origin $\mathcal{O} \in M$. If we use Cartesian coordinates x^i with the origin at $\mathcal{O}$ (thus, all x^i vanish at $\mathcal{O}$), then these quantities are analytically given by the formulas

$$\Lambda^i_j := x^i p_j, \qquad \mathcal{I}^i_j := \Lambda^i_j + \Sigma^i_j \tag{69}$$

where Λ^i_j are Hamiltonian generators of the centre-affine group acting in M and leaving $\mathcal{O}$ invariant. For any $A \in GL(V)$, these transformations act as follows: the point $x \in M$ with coordinates x^i is transformed onto one with coordinates $A^i_j x^j$; the internal configuration φ is non-affected. Similarly, $\mathcal{I}^i_j$ are Hamiltonian generators of this group acting in the total configuration space. This action of $GL(V)$ is given by: $(x^i, \varphi^i_K) \mapsto (A^i_j x^j, A^i_j \varphi^j_K)$ for any $A \in GL(V)$. The doubled g-skew-symmetric parts of $\Lambda, \mathcal{I}$ are respectively the translational and total angular momentum with respect to $\mathcal{O} \in M$; we denote these quantities by L, $\mathcal{J}$. Obviously, $\mathcal{J} = L + S$.

Finally, let us quote a few fundamental Poisson brackets. This is the special case of (24), (25) corresponding to the affine group, however reformulated in geometric terms of the physical and material space:

$$\begin{gathered}\{\Sigma^i_j, \Sigma^k_l\} = \delta^i_l \Sigma^k_j - \delta^k_j \Sigma^i_l, \text{ and similarly for } \Lambda^i_j, \mathcal{I}^i_j \\ \{\Lambda^i_j, \Sigma^k_l\} = 0, \quad \{\hat{\Sigma}^A_B, \hat{\Sigma}^C_D\} = \delta^C_B \hat{\Sigma}^A_D - \delta^A_D \hat{\Sigma}^C_B, \quad \{\Sigma^i_j, \hat{\Sigma}^A_B\} = 0 \\ \{\hat{\Sigma}^A_B, \hat{p}_C\} = -\delta^A_C \hat{p}_B, \quad \{\mathcal{I}^k_j, p_i\} = \{\Lambda^k_j, p_i\} = \delta^k_i p_j.\end{gathered} \tag{70}$$

One easily recognizes here the $G\,\mathrm{Aff}(n, \mathbb{R})$ and $GL(n, \mathbb{R})$ structure constants. Obviously, $\hat{p}_A$ are co-moving components of translational canonical momenta

$$\hat{p}_A = p_i \varphi^i{}_A, \qquad \hat{p} = \varphi^* p. \tag{71}$$

Besides, for functions F depending only on the configuration variables (x, φ) we have, as the special case of

$$\{\Sigma^i_j, F\} = -\varphi^i{}_A \frac{\partial F}{\partial \varphi^j{}_A}, \quad \{\Lambda^i_j, F\} = -x^i \frac{\partial F}{\partial x^j}, \quad \{\hat{\Sigma}^A_B, F\} = -\varphi^i{}_B \frac{\partial F}{\partial \varphi^i{}_A}. \tag{72}$$

Using these formulas we shall write and analyze equations of motion in the Poisson-Hamilton form,

$$\frac{\mathrm{d}G}{\mathrm{d}t} = \{G, H\}. \tag{73}$$

In practical problems G runs over some maximal family of functionally independent functions on the phase space manifold.

4. Dynamical Affine Invariance of Geodetic Systems

At least for purely academic reasons it is interesting to construct kinetic energy models (metric tensors on the configuration space) invariant under the spatial (left-acting) or material (right-acting) affine groups. The resulting models belong to the class of invariant geodetic systems with Lie-group-ruled degrees of freedom, as investigated, e.g. by Arnold [3]. And the right-affinely-invariant models may be considered as a very drastic discretization, reduction to a finite number of degrees of freedom of the Arnold description of the ideal fluid as an infinite-dimensional Hamiltionian system on the group $\mathrm{SDiff}(n, \mathbb{R})$ of all volume-preserving diffeomorphisms.

The corresponding kinetic energies are not based on the d'Alembert principle (constraints-restriction of the usual multi-particle energy) but only on the appropriate invariance demands. Some of such invariant geodetic models may describe elastic-like bounded vibrations even without any extra-introduced potential. In a sense, interactions are encoded in the used metric tensor on the configuration space, just as, e.g., in Maupertuis variational principle. Physically, it may be expected that models of this kind may describe the collective nuclear dynamics. There are no reasons to expect the d'Alembert model to work there. D'Alembert principle and the constraints-restricted kinetic energy model work when the collective motion is a "large" background perturbed by small, negligible non-collective vibrations. In the droplet description of nuclei, the underlying non-collective micro-motion may be just "large", and collective modes may appear as some average kinematical characteristic of this hidden microscopic motion. Then the usual d'Alembert mechanism does not work and the simplest procedure is to postulate some phenomenological model on the basis of invariance principles. Other physical applications may be expected in the theory of defects in solids and perhaps in dynamics of some macroscopic objects, like, e.g., gas bubbles in fluids.

Let us now describe kinetic energies (metric tensors on Q) affinely-invariant in M (spatially-, i.e. left-invariant). When the configuration space is simply identified with a Lie group, the corresponding general formula is given by (15). Explicitly, it becomes now

$$T = T_{\mathrm{tr}} + T_{\mathrm{int}} \tag{74}$$

where the internal and translational parts are respectively given by:

$$T_{\mathrm{int}} = \frac{1}{2}\mathcal{L}^{BD}_{AC}\hat{\Omega}^{A}_{B}\hat{\Omega}^{C}_{D}, \quad T_{\mathrm{tr}} = \frac{m}{2}C_{ij}\frac{\mathrm{d}x^{i}}{\mathrm{d}t}\frac{\mathrm{d}x^{j}}{\mathrm{d}t} = \frac{m}{2}\eta_{AB}\hat{v}^{A}\hat{v}^{B} \tag{75}$$

where $\mathcal{L}^{BD}_{AC} = \mathcal{L}^{DB}_{CA} = \mathrm{const}$.

Legendre transformation in the translational and internal sectors has respectively the form:

$$p_{i} = \frac{\partial T}{\partial \dot{x}^{i}} = mC_{ij}(\varphi)\frac{\mathrm{d}x^{j}}{\mathrm{d}t}, \qquad \hat{\Sigma}^{A}_{B} = \frac{\partial T}{\partial \hat{\Omega}^{B}_{A}} = \mathcal{L}^{AC}_{BD}\hat{\Omega}^{D}_{C}. \tag{76}$$

In certain calculations it is more convenient to use the representation

$$\Sigma^{i}_{j} = \frac{\partial T}{\partial \Omega^{j}_{i}} = \tilde{\mathcal{L}}^{ik}_{jl}\Omega^{l}_{k} \tag{77}$$

where $\tilde{\mathcal{L}}^{ik}_{jl} = \varphi^{i}{}_{A}\varphi^{-1B}{}_{j}\varphi^{k}{}_{D}\varphi^{-1C}{}_{l}\mathcal{L}^{AD}_{BC}$. Obviously, the same form is true for the potential-type Lagrangians $L = T - V(x, \varphi)$. In the geodetic case the Noether theorem implies that both the translational momentum p_{i} and the total affine momentum $\mathcal{I}^{i}_{j} = x^{i}p_{j} + \Sigma^{i}_{j}$ are constants of motion.

Remark: p_{i} is a constant of motion but $\mathrm{d}x^{i}/\mathrm{d}t$ is not, because they are interrelated through the φ-dependent Cauchy deformation tensor. Even the direction of v^{i} is in general variable. This phenomenon could be called the "drunk missile effect".

It is important that in the above model of T the metric tensor $g \in V^{*} \otimes V^{*}$ does not occur at all and the physical space may be considered as a purely affine, amorphous (metric-free) space.

Kinetic energies (metric tensors on Q) affinely-invariant in N (materially-, i.e. right-invariant) $T = T_{\rm tr} + T_{\rm int}$ are given by

$$T_{\rm int} = \frac{1}{2}\Re^{jl}_{ik}\Omega^i_j\Omega^k_l, \quad T_{\rm tr} = \frac{m}{2}g_{ij}\frac{\mathrm{d}x^i}{\mathrm{d}t}\frac{\mathrm{d}x^j}{\mathrm{d}t} = \frac{m}{2}G_{AB}\hat{v}^A\hat{v}^B \tag{78}$$

where $\Re^{jl}_{ik} = \Re^{lj}_{ki} = \mathrm{const}$.

For geodetic and potential systems Legendre transformation has the form

$$p_i = \frac{\partial T}{\partial \dot{x}^i} = mg_{ij}\frac{\mathrm{d}x^j}{\mathrm{d}t}, \qquad \Sigma^i_j = \frac{\partial T}{\partial \Omega^j_i} = \Re^{ik}_{jl}\Omega^l_k. \tag{79}$$

When studying the equations of motion it is convenient to use the form

$$\hat{\Sigma}^A_B = \frac{\partial T}{\partial \hat{\Omega}^B_A} = \tilde{\Re}^{AC}_{BD}\hat{\Omega}^D_C \tag{80}$$

where $\tilde{\Re}^{AC}_{BD} = \varphi^{-1}{}^A_i\varphi^j_B\varphi^{-1}{}^C_k\varphi^l_D\Re^{ik}_{jl}$.

Remark: Strictly speaking, one should use the term "linear material invariance" instead of "affine material invariance". The reason is that there is no translational invariance in N; the centre of mass is fixed there and reduces the symmetry to the centre-affine one, isomorphic with $GL(U)$.

Constraining internal configurations φ to the volume-preserving ones, one obtains something like the discretization of the SDiff$(n, \mathbb{R})$-based Arnold theory of ideal fluids.

Let us write down explicitly equations of motion as balance laws for the group generators. We admit the potential terms in Lagrangians, $L = T - \mathcal{V}(x, \varphi)$.

For systems with the $\pounds$-type kinetic energies, $L = T_\pounds - \mathcal{V}(x, \varphi)$, we obtain, using the Poisson-bracket method, the following balance equations:

$$\frac{\mathrm{d}p_i}{\mathrm{d}t} = Q_i, \qquad \frac{\mathrm{d}\Sigma^i_j}{\mathrm{d}t} = -\frac{1}{m}\tilde{C}^{ik}p_kp_j + Q^i_j \tag{81}$$

where generalized forces are given by

$$Q_i = -\frac{\partial \mathcal{V}}{\partial x^i}, \qquad Q^i_j = -\varphi^i{}_A\frac{\partial \mathcal{V}}{\partial \varphi^j{}_A} \tag{82}$$

or, alternatively,

$$\frac{\mathrm{d}p_i}{\mathrm{d}t} = Q_i, \qquad \frac{\mathrm{d}\mathcal{I}^i_j}{\mathrm{d}t} = Q_{\rm tot}{}^i_j = x^iQ_j + Q^i_j. \tag{83}$$

In this form these equations may be easily generalized to ones describing dissipative systems.

When taken together with the above Legendre transformation, these equations become a closed system of equations of motion. For geodetic systems, when $Q_i = 0$, $Q^i_j = 0$, they reduce simply to the conservation laws for p_i, $\mathcal{I}^i_j$, which are affine group generators in M. This is simply the consequence of the affine invariance in M (application of the Noether theorem). Because of this, it is better to use the balance laws for Σ^i_j than those for $\hat{\Sigma}^A_B$. Kinetic energy is simple when expressed through $\hat{\Omega}^A_B$ or in terms of phase-space variables through $\hat{\Sigma}^A_B$; it is then a constant-coefficients quadratic form of $\hat{\Omega}$ or $\hat{\Sigma}$. However, to obtain conservation laws one should use the spatial tensors Σ^i_j and $\mathcal{I}^i_j$. And then, Legendre transformation has a rather complicated form, whereas with the use of material variables $\hat{\Sigma}^A_B$ it was simple. But the balance for $\hat{\Sigma}^A_B$ is complicated and non-readable.

For systems with the $\Re$-type kinetic energies, $L = T_{\Re} - \mathcal{V}(x, \varphi)$, we have the following balance form of dynamical equations:

$$\frac{\mathrm{d}p_a}{\mathrm{d}t} = Q_a, \qquad \frac{\mathrm{d}\hat{\Sigma}^A_B}{\mathrm{d}t} = \hat{Q}^A_B \tag{84}$$

where

$$Q_a = -\frac{\partial \mathcal{V}}{\partial x^a}, \quad \hat{Q}^A_B = -\frac{\partial \mathcal{V}}{\partial \varphi^i_A}\varphi^i_B = \varphi^{-1A}{}_i Q^i_j \varphi^j_B. \tag{85}$$

And again this system of equations is closed when considered jointly with Legendre transformations. The latter ones are simple when considered in spatial terms, i.e. with the use of Ω^i_j, Σ^i_j. However, the balance laws for Σ^i_j have a rather complicated form. And those for $\hat{\Sigma}^A_B$ are simple and reduce to the conservation laws when the system is geodetic, i.e. $\mathcal{V} = 0$.

Let us stress an important point: there are no geodetic models affinely-invariant (amorphous) simultaneously in the physical space M and in the material space N. This is because of the malicious non-semisimplicity of $G\,\mathrm{Aff}(n, \mathbb{R}) \simeq GL(n, \mathbb{R}) \times_s \mathbb{R}^n$. Any twice covariant tensor field on $G\,\mathrm{Aff}(n, \mathbb{R})$, which is simultaneously left- and right-invariant, must be degenerate. The highest possible symmetries compatible with the non-singularity demand for the metric are the following:

- affine symmetry in the space M and metrical (Euclidean) symmetry in the material space N. Then we do not need any physical metric $g \in V^* \otimes V^*$ at all.
- conversely: Euclidean symmetry in the physical space M and affine symmetry in the material space N. Then we do not need any material metric $\eta \in U^* \otimes U^*$ at all.

If we neglect translational motion, the situation changes drastically. There exist models invariant simultaneously under $GL(V)$ and $GL(U)$. In other words, there exist Hamiltonian geodetic systems on $GL(n, \mathbb{R})$ invariant simultaneously under left and right regular translations; the underlying metrics are non-degenerate. However, they are never positively-definite (but they may be physically applicable, e.g. in the theory of one-dimensional lattices). This argument, together with the previous one, fixes our attention on models on $Q_{\rm int} = LI(U,V)$ invariant under $GL(V) \times SO(U,\eta)$ or $SO(V,g) \times GL(U)$. Such geodetic models may have positively-definite kinetic energy.

5. Affine Systems Without Translational Motion. Geodetic Models

Let us now forget about "trivial" translational degrees of freedom and concentrate on geometry of $Q_{\rm int} = LI(U,V)$ or its other representations like $F(V)$ or $GL(n,\mathbb{R})$ (corresponding respectively to the choices: $U = \mathbb{R}^n$ or $U = V = \mathbb{R}^n$). Spatially and materially affine geodetic systems correspond respectively to the above $\mathfrak{L}$- and $\mathfrak{R}$-models of $T_{\rm int}$. The most general geodetic models invariant simultaneously under $GL(V)$ and $GL(U)$ are given by:

$$T_{\rm int} = \frac{A}{2}\,\mathrm{Tr}(\Omega^2) + \frac{B}{2}(\mathrm{Tr}\,\Omega)^2 = \frac{A}{2}\,\mathrm{Tr}(\hat{\Omega}^2) + \frac{B}{2}(\mathrm{Tr}\,\hat{\Omega})^2. \tag{86}$$

The first term has the hyperbolic signature

$$\left(\frac{1}{2}n(n+1)+, \frac{1}{2}n(n-1)-\right) \tag{87}$$

where, roughly speaking, the plus signs correspond to the non-compact dimensions of $GL(n,\mathbb{R})$, and the minus signs to the compact ones. If (and only if) $A = -nB$, the underlying metric on $Q_{\rm int}$ becomes degenerate. The singular direction corresponds to the dilatational centre. In particular, the special choice $A = 2n$, $B = -2$ is just the standard normalization of the Killing metric.

Legendre transformation may be described in two equivalent ways:

$$\Sigma = A\Omega + B(\mathrm{Tr}\Omega)I_V, \qquad \hat{\Sigma} = A\hat{\Omega} + B(\mathrm{Tr}\hat{\Omega})I_U \tag{88}$$

where I_V and I_U denote the identity transformations of V and U, respectively. The resulting geodetic Hamiltonians are given by

$$\mathfrak{T} = \frac{1}{2A}\,\mathrm{Tr}(\Sigma^2) - \frac{B}{2A(A+nB)}(\mathrm{Tr}\,\Sigma)^2 \tag{89}$$

or, analogously, with the use of $\hat{\Sigma}$ instead of Σ. General solution of the corresponding equations of geodetic motion has the form

$$\varphi(t) = \mathrm{e}^{Et}\varphi_0 = \varphi_0\mathrm{e}^{\hat{E}t} \tag{90}$$

where $\varphi_0 \in LI(U,V)$ is arbitrary, $E \in L(V) \simeq \mathfrak{gl}(V)$ is arbitrary, $\hat{E} \in L(U) \simeq \mathfrak{gl}(U)$, $\hat{E} = \varphi_0^{-1}E\varphi_0$, is also arbitrary. The pairs (φ_0, E) and $(\varphi_0, \hat{E})$ are alternative descriptions of initial conditions, because

$$\varphi_0 = \varphi(0), \quad E = \Omega(0), \quad \hat{E} = \hat{\Omega}(0), \quad \dot{\varphi}(0) = E\varphi_0 = \varphi_0\hat{E}. \tag{91}$$

Obviously, Σ and $\hat{\Sigma}$ or, alternatively, Ω and $\hat{\Omega}$ are constants of motion.

This is analogous to the general solution for the spherical (metrically) rigid body. A and B play the role of generalized constant moments of inertia.

When we identify U and V with $\mathbb{R}^n$ and $LI(U,V)$ with $GL(n,\mathbb{R})$, the above formulas mean that the general solution consists of all one-parameter subgroups and their cosets (does not matter, left or right; any right coset is a left coset with respect to another subgroup). Now, after having defined everything in terms of geometric objects in M and N spaces we return temporarily to the analytical description based on the mentioned identification. This is done for analytical simplicity, and everything may be easily translated into geometrical language.

Thus, we again identify

$$Q_{\text{int}} \simeq GL^{+}(n,\mathbb{R}) = \mathbb{R}^{+} \otimes SL(n,\mathbb{R}) = \mathrm{e}^{\mathbb{R}} \otimes SL(n,\mathbb{R}). \tag{92}$$

It will be convenient to factorize configurations φ into dilatational and isochoric (incompressible) terms:

$$GL^{+}(n,\mathbb{R}) \ni \varphi = l\Psi = \mathrm{e}^{q}\Psi, \qquad \Psi \in SL(n,\mathbb{R}). \tag{93}$$

Using the continuum language, Ψ describes rotational and shear motion. Let us denote

$$\omega := \frac{\mathrm{d}\Psi}{\mathrm{d}t}\Psi^{-1}, \qquad \hat{\omega} := \Psi^{-1}\frac{\mathrm{d}\Psi}{\mathrm{d}t}. \tag{94}$$

Then

$$\Omega = \omega + \frac{\mathrm{d}q}{\mathrm{d}t}I, \qquad \hat{\Omega} = \hat{\omega} + \frac{\mathrm{d}q}{\mathrm{d}t}I, \qquad \Sigma = \sigma + \frac{p}{n}I \qquad \hat{\Sigma} = \hat{\sigma} + \frac{p}{n}I \tag{95}$$

where p is the conjugate momentum of q, $\{q,p\} = 1$, $\omega, \hat{\omega}, \sigma, \hat{\sigma} \in \mathfrak{sl}(n)$, i.e. they are trace-less, and I denotes the identity matrix. Geometrically, in terms of V, U-spaces, $\omega \in \mathfrak{sl}(V)$, $\sigma \in \mathfrak{sl}(V)^{*} \simeq \mathfrak{sl}(V)$, $\hat{\omega} \in \mathfrak{sl}(U)$, $\hat{\sigma} \in \mathfrak{sl}(U)$, and as previously Ω, Σ are elements of $L(V)$ and $\hat{\Omega}$, $\hat{\Sigma}$ belong to $L(U)$. The identity matrix I is to be replaced by I_V and I_U respectively in expressions for (Ω,Σ) and $(\hat{\Omega},\hat{\Sigma})$. The pairing between momenta and velocities is given by

$$\mathrm{Tr}(\Sigma\Omega) = \mathrm{Tr}(\hat{\Sigma}\hat{\Omega}) = \mathrm{Tr}(\sigma\omega) + p\dot{q} = \mathrm{Tr}(\hat{\sigma}\hat{\omega}) + p\dot{q}. \tag{96}$$

The two-side invariant kinetic energy may be written down as follows

$$T = \frac{A}{2}\mathrm{Tr}(\omega^2) + \frac{1}{2}(A+Bn)\dot{q}^2 = T_{\text{sh}} + T_{\text{dil}}. \tag{97}$$

In this formula ω may be replaced by $\hat{\omega}$.

Legendre transformation takes on the form

$$\sigma = A\omega \quad \text{or} \quad \hat{\sigma} = A\hat{\omega}, \qquad p = n(A+Bn)\dot{q}. \tag{98}$$

The resulting geodetic Hamiltonian $\Im = \Im_{\text{sh}} + \Im_{\text{dil}}$ has the form

$$\Im = \frac{1}{2A}\mathrm{Tr}(\sigma^2) + \frac{1}{2n(A+Bn)}p^2 = \frac{1}{2A}\mathrm{Tr}(\hat{\sigma}^2) + \frac{1}{2n(A+Bn)}p^2. \tag{99}$$

In geodetic models, dilatational motion is unbounded, except the solution $l \equiv 1$, i.e. $q \equiv 0$. But this solution is exponentially non-stable on the level of l (although the squeezing to $l = 0$ needs the infinite time). Therefore, one must either assume the incompressibility constraints $q = 0$ or introduce some stabilizing dilatational potential

$$\begin{aligned} L &= L_{\rm sh} + L_{\rm int} = T_{\rm sh} + T_{\rm int} - \mathcal{V}(q) \\ H &= H_{\rm sh} + H_{\rm int} = \Im_{\rm sh} + \Im_{\rm int} + \mathcal{V}(q). \end{aligned} \tag{100}$$

The shear-rotational ($SL(n, \mathbb{R})$) and dilatational motions are in such models completely separable and independent. More generally, this is true for explicitly separable potentials,

$$\mathcal{V}(\varphi) = \mathcal{V}(\Psi, q) = \mathcal{V}_{\rm sh}(\Psi) + \mathcal{V}_{\rm dil}(q). \tag{101}$$

What concerns the possible shapes of the stabilizing dilatational term $\mathcal{V}_{\rm dil}(q)$, it is reasonable to use simple phenomenological models, like, e.g., the harmonic oscillator,

$$\mathcal{V}_{\rm dil}(q) = \frac{\kappa}{2} q^2 \tag{102}$$

or some infinite potential well with elastically reflecting walls,

$$\mathcal{V}_{\rm dil}(q) = \infty \quad \text{for} \quad |q| > d, \qquad \mathcal{V}_{\rm dil}(q) = 0, \quad \text{for} \quad |q| < d. \tag{103}$$

One can also use finite potential well,

$$\mathcal{V}_{\rm dil}(q) = 0 \quad \text{for} \quad |q| > d, \qquad \mathcal{V}_{\rm dil}(q) = c < 0, \quad \text{for} \quad |q| < d. \tag{104}$$

These well models are particularly convincing in quantum problems.

Let us now consider the geodetic models on $SL(n, \mathbb{R})$. The number of degrees of freedom equals $(n^2 - 1) = \dim SL(n, \mathbb{R})$. We are interested in models describing elastic, bounded vibrations. The fundamental question is the following:

- Does a $2(n^2 - 1)$-dimensional family of bounded solutions exist? (below-dissociation-threshold situations)
- Does a $2(n^2 - 1)$-dimensional family of non-bounded, escaping solutions exist? (above-dissociation-threshold situations)

The answer is affirmative. Let us present an outline of the reasoning supporting the statement that there is an open family of bounded and an open family of escaping solutions within the general solution of the doubly-invariant geodetic problem on $SL(n, \mathbb{R})$.

Let $\alpha \in \mathfrak{sl}(n)$ ($\operatorname{Tr} \alpha = 0$) be similar to an antisymmetric matrix $\lambda = -\lambda^T \in \mathfrak{so}(n)$, $\alpha = \chi \lambda \chi^{-1}$ for some $\chi \in SL(n, \mathbb{R})$. Then every motion

$$\Psi(t) = \mathrm{e}^{\alpha t} \Psi_0 = \chi \mathrm{e}^{\lambda t} \chi^{-1} \Psi_0 \tag{105}$$

is bounded. The structure constants (simplicity of $SL(n,\mathbb{R})$) imply that the set of such α-s is (n^2-1)-dimensional, although $\dim SO(n,\mathbb{R}) = n(n-1)/2$. Nevertheless, it is not so that these n^2-1 velocity parameters combine additively with n^2-1 parameters of Ψ_0 so as to result in $2(n^2-1)$ parameters (initial conditions) in the phase space. The reason is that appropriate correlations between Ψ_0 and λ may repeat the same orbits. In dimensions $n = 2, 3$ the above solutions are always periodic. In higher dimensions they may be so but need not. Take for example $n = 4$, represent $\mathbb{R}^4$ as $\mathbb{R}^2 \times \mathbb{R}^2$ and assume that λ is a block matrix consisting of two 2×2 skew-symmetric blocks. Any of these blocks has essentially one parameter. If the ratio of these angular velocity parameters is an irrational number, then the resulting motion is non-periodic, its orbit is not closed and because of this it is not a Lie subgroup in the usual sense, although it is an algebraic subgroup. The closures of such orbits are two-dimensional submanifolds. But one can also show that there are bounded non-periodic solutions in two and three dimensions as well. The point is that the mentioned matrices λ may be slightly perturbed by small symmetric matrices κ and we can take the solutions

$$\Psi(t) = \chi \mathrm{e}^{(\lambda+\kappa)t}\chi^{-1}. \tag{106}$$

The afore-mentioned periodic orbits (corresponding to $\kappa = 0$) are stable in the sense that for some open range of $\kappa = \kappa^T \in \mathfrak{sl}(n)$, i.e. for some open range of $\alpha = \lambda + \kappa \in \mathfrak{sl}(n)$ the resulting motion is still bounded although no longer periodic. And there is sufficiently much of the above matrices α so as not to interfere with the arbitrariness of Ψ_0. Thus, the corresponding family of solutions contains an open subset (in the sense of initial conditions) of the general solution.

Quite similarly, if we took symmetric $\lambda = \lambda^T \in \mathfrak{sl}(n)$, then the corresponding solutions $\Psi(t) = \chi \mathrm{e}^{\lambda t}\chi^{-1}\Psi_0 = \mathrm{e}^{\chi\lambda\chi^{-1}t}\Psi_0$ would be non-bounded (escaping). And it will be so if we slightly perturb λ by "small" antisymmetric matrices $\epsilon = -\epsilon \in \mathfrak{so}(n)$ from some open neighbourhood of the null element. And again we conclude that the general solution contains an open subset of unbounded (escaping) trajectories.

The quantum counterpart is obvious: In quantized geodetic models there exists a discrete energy spectrum of physically bounded situations, and above it – the continuous spectrum corresponding to the dissociated body. There is an obvious analogy with the $E < 0$ and $E > 0$ situations for the Coulomb problem.

Remark: When using geometric terms, we should replace the word "skew-symmetric" by "g-skew-symmetric" or "η-skew-symmetric", and similarly with "symmetric":

$$\lambda^i{}_j = \mp g^{ik} g_{mj} \lambda^m{}_k, \qquad \hat{\lambda}^A{}_B = \mp \eta^{AC}\eta_{DB}\hat{\lambda}^D{}_C. \tag{107}$$

Now let us go back to models invariant under $GL(V) \times SO(U,\eta)$ and $SO(V,g) \times GL(U)$.

We begin with affine-metrical models, i.e. ones invariant under $GL(V)\times SO(U,\eta)$. The spatial metric tensor $g\in V^{*}\otimes V^{*}$ does not occur in any formulas and because of this it need not exist at all; the physical space may be purely affine one. Now we have three scalars of generalized inertial momentum I, A, B, and $T_{\rm int}$ is given by:

$$T_{\rm int}=\frac{I}{2}\eta_{KL}\eta^{MN}\hat{\Omega}^{K}_{M}\hat{\Omega}^{L}_{N}+\frac{A}{2}\,{\rm Tr}(\hat{\Omega}^{2})+\frac{B}{2}({\rm Tr}\,\hat{\Omega})^{2}. \qquad (108)$$

One can show that unlike the doubly-invariant $T_{\rm int}$ (86), this expression is positively definite in some open range of triples (I,A,B).

In matrix terms, when $\eta_{KL}=\delta_{KL}$, we have:

$$T_{\rm int}=\frac{I}{2}{\rm Tr}(\hat{\Omega}^{T}\hat{\Omega})+\frac{A}{2}{\rm Tr}(\hat{\Omega}^{2})+\frac{B}{2}({\rm Tr}\hat{\Omega})^{2}. \qquad (109)$$

Obviously, in the second and third terms, $\hat{\Omega}$ may be replaced by Ω. Legendre transformation has the form:

$$\hat{\Sigma}^{K}_{L}=I\eta^{KM}\eta_{LN}\hat{\Omega}^{N}_{M}+A\hat{\Omega}^{K}_{L}+B\hat{\Omega}^{M}_{M}\delta^{K}_{L}. \qquad (110)$$

In matrix terms, when $\eta_{KL}=\delta_{KL}$, we have

$$\hat{\Sigma}=I\hat{\Omega}^{T}+A\hat{\Omega}+B({\rm Tr}\,\hat{\Omega})I_{n} \qquad (111)$$

where I_n denotes the $n\times n$ identity matrix. Another convenient form is

$$\Sigma^{i}_{j}=I\tilde{C}^{ib}C_{ja}\Omega^{a}_{b}+A\Omega^{i}_{j}+B\Omega^{m}_{m}\delta^{i}_{j}. \qquad (112)$$

The corresponding geodetic Hamiltonian is given by

$$\Im_{\rm int}=\frac{1}{2\tilde{I}}\eta_{KL}\eta^{MN}\hat{\Sigma}^{K}_{M}\hat{\Sigma}^{L}_{N}+\frac{1}{2\tilde{A}}\hat{\Sigma}^{K}_{L}\hat{\Sigma}^{L}_{K}+\frac{1}{2\tilde{B}}\hat{\Sigma}^{K}_{K}\hat{\Sigma}^{L}_{L} \qquad (113)$$

where

$$\tilde{I}=\frac{1}{I}(I^{2}-A^{2}),\quad \tilde{A}=\frac{1}{A}(A^{2}-I^{2}),\quad \tilde{B}=-\frac{1}{B}(I+A)(I+A+nB). \qquad (114)$$

Sometimes it is convenient to use another equivalent form

$$\Im_{\rm int}=\frac{1}{2\tilde{I}}C_{kl}\tilde{C}^{mn}\Sigma^{k}_{m}\Sigma^{l}_{n}+\frac{1}{2\tilde{A}}\Sigma^{k}_{l}\Sigma^{l}_{k}+\frac{1}{2\tilde{B}}\Sigma^{k}_{k}\Sigma^{l}_{l}. \qquad (115)$$

Using the analytical matrix terms we can write

$$\Im_{\rm int}=\frac{1}{2\tilde{I}}\,{\rm Tr}(\hat{\Sigma}^{T}\hat{\Sigma})+\frac{1}{2\tilde{A}}\,{\rm Tr}(\hat{\Sigma}^{2})+\frac{1}{2\tilde{B}}({\rm Tr}\,\hat{\Sigma})^{2}. \qquad (116)$$

The most convenient representation is the following:

$$\Im_{\rm int}=\frac{1}{2\alpha}C(2)+\frac{1}{2\beta}C(1)^{2}+\frac{1}{2\mu}\|V\|^{2} \qquad (117)$$

where $C(2)$, $C(1)$ are respectively the second-order and first-order Casimir invariants

$$C(2)={\rm Tr}(\Sigma^{2})={\rm Tr}(\hat{\Sigma}^{2}),\quad C(1)={\rm Tr}(\Sigma)={\rm Tr}(\hat{\Sigma})=p \qquad (118)$$

$\|V\|$ is the scalar magnitude of vorticity,

$$\|V\|^2 = -\frac{1}{2}\operatorname{Tr}(V^2) \tag{119}$$

and α, β, μ are given by

$$\alpha = I + A, \quad \beta = -\frac{1}{B}(I + A)(I + A + nB), \quad \mu = \frac{1}{I}(I^2 - A^2). \tag{120}$$

Just this form implies that for $H = \Im_{\text{int}} + \mathcal{V}_{\text{dil}}(q)$ there exists also an open family of bounded solutions. Moreover, the same is true for any Hamiltonian depending on the deformation invariants $K = (K_1, \ldots, K_n)$

$$H = \Im_{\text{int}} + \mathcal{V}(K_1, \ldots, K_n) \tag{121}$$

provided that $\mathcal{V}$ depends on K in such a way that dilatational vibrations and vibrations in all remaining deformation invariants are stabilized (q is a function of deformation invariants or simply may be chosen as one of basic invariants).

We are however particularly interested in models $H = \Im_{\text{int}} + \mathcal{V}(q)$ geodetic in $SL(n,\mathbb{R})$ and stabilized in dilatational vibrations. The above statements follow from the fact that the evolution of variables Σ, K is the same for the Hamiltonian H with $\Im_{\text{int}}$ given by (117) and for the Hamiltonian H with the two-side affinely-invariant $\Im_{\text{int}}$ corresponding to the choice $I = 0$, i.e. $1/\mu = 0$. This follows from the Poisson-bracket form of equations of motion. Indeed, $\|V\|^2$ is a constant of motion for both types of Hamiltonians (i.e. for those with $I = 0$ and $I \neq 0$). The quantities Σ, K, $\|V\|$ have the following Poisson brackets in addition to those Lie-algebraic ones satisfied by $\Sigma^i{}_j$

$$\{\Sigma^i{}_j, C(2)\} = \{\Sigma^i{}_j, C(1)\} = 0, \quad \{\Sigma^i{}_j, \|V\|^2\} = 0, \quad \{K_a, \|V\|^2\} = 0. \tag{122}$$

The last formula follows from the fact that the deformation invariants are invariant under right regular translations by elements of $SO(n,\mathbb{R})$, generated by $V^A{}_B$. The first formula holds just because $C(2)$, $C(1)$ are Casimir invariants of $\Sigma^i{}_j$ (and $\hat{\Sigma}^A{}_B$). The second formula is due to equations $\{\Sigma^i{}_j, \hat{\Sigma}^A{}_B\} = 0$ ($\|V\|^2$ is algebraically built of $\hat{\Sigma}^A{}_B$). Obviously,

$$\{K_a, \Sigma^i{}_j\} \neq 0, \quad \{K_a, C(2)\} \neq 0, \quad \{K_a, C(1)\} \neq 0 \tag{123}$$

but they have the same form for $I = 0$ ($1/\mu = 0$) and $I \neq 0$ ($1/\mu \neq 0$), because $\{K_a, \|V\|^2\} = 0$. Therefore, on the level of variables $\Sigma^i{}_j$, K_a, the model with the affine-metric kinetic energy ($GL(V) \times SO(U,\eta)$-invariant) looks identical with the one based on affine-affine ($GL(V) \times GL(U)$-invariant) Casimir kinetic term. Evolution of variables $\Sigma^i{}_j$, K_a is identical in both models, and everything stated above about bounded motions and escaping trajectories in Casimir models remains true for the affine-metrical T-model. The only difference appears in degrees of freedom ruled by the $SO(V, g)$- and $SO(U, \eta)$-groups. These degrees of freedom are equivalent to orientations of the principal axes of the Cauchy and Green deformation tensors (with respect to the metrics g, η). But these degrees of freedom have the compact-manifold topology, thus their evolution does not influence the bounded or non-bounded character of trajectories (the evolution of compact-manifold-variables is always bounded). In particular, for models with the stabilized dilatations and geodetic $SL(n, \mathbb{R})$-dynamics everything stated above remains true.

It is instructive to express the affine-metric $\Im_{\rm int}$ in terms of the $SL(n, \mathbb{R})$-dilatations splitting. One can show that

$$\Im_{\rm int} = \frac{1}{2(I+A)} C_{SL(n)}(2) + \frac{1}{2n(I+A+nB)} p^2 + \frac{I}{2(I^2-A^2)} |||V||^2 \quad (124)$$

where $C_{SL(n)}(2) = {\rm Tr}(\sigma^2) = {\rm Tr}(\hat{\sigma}^2)$. From this expression it is easily seen that there exists an open subset of bounded and an open subset of non-bounded trajectories within the general solution for geodetic incompressible models.

Let us now repeat the same for the metrical-affine models, i.e. for kinetic energies invariant under $SO(V, g) \times GL(U)$. Physical space is Euclidean with the metric $g \in V^* \otimes V^*$, whereas the material one is purely affine and no metric $\eta \in U^* \otimes U^*$ is assumed. $T_{\rm int}$ is given by

$$T_{\rm int} = \frac{I}{2} g_{ik} g^{jl} \Omega^i{}_j \Omega^k{}_l + \frac{A}{2} {\rm Tr}(\Omega^2) + \frac{B}{2} ({\rm Tr}\, \Omega)^2 \quad (125)$$

where I, A, B are generalized scalar inertial moments as above, and for some open range of (I, A, B) $T_{\rm int}$ may be positively-definite. In matrix terms, when $g_{ij} = \delta_{ij}$, we have

$$T_{\rm int} = \frac{I}{2} {\rm Tr}(\Omega^T \Omega) + \frac{A}{2} {\rm Tr}(\Omega^2) + \frac{B}{2} ({\rm Tr}\, \Omega)^2. \quad (126)$$

Legendre transformation has the form:

$$\Sigma^i{}_j = I g^{im} g_{jn} \Omega^n{}_m + A \Omega^i{}_j + B \Omega^m{}_m \delta^i{}_j. \quad (127)$$

In matrix terms, when $g_{ij} = \delta_{ij}$,

$$\Sigma = I \Omega^T + A \Omega + B ({\rm Tr}\, \Omega) I_n. \quad (128)$$

Therefore, the geodetic Hamiltonian has the form

$$\Im_{\rm int} = \frac{1}{2\tilde{I}} g_{ik} g^{jl} \Sigma^i{}_j \Sigma^k{}_l + \frac{1}{2\tilde{A}} \Sigma^i{}_j \Sigma^j{}_i + \frac{1}{2\tilde{B}} \Sigma^i{}_i \Sigma^j{}_j \quad (129)$$

with the same meaning of constants as previously.

Let us quote, as previously, other equivalent representations, e.g.

$$\Im_{\rm int} = \frac{1}{2\tilde{I}} G_{KL} \tilde{G}^{MN} \hat{\Sigma}^K{}_M \hat{\Sigma}^L{}_N + \frac{1}{2\tilde{A}} \hat{\Sigma}^K{}_L \hat{\Sigma}^L{}_K + \frac{1}{2\tilde{B}} \hat{\Sigma}^K{}_K \hat{\Sigma}^L{}_L \quad (130)$$

or in matrix terms

$$\Im_{\rm int} = \frac{1}{2\tilde{I}}\,{\rm Tr}(\hat{\Sigma}^R\hat{\Sigma}) + \frac{1}{2\tilde{A}}\,{\rm Tr}(\hat{\Sigma}^2) + \frac{1}{2\tilde{B}}({\rm Tr}\,\hat{\Sigma})^2. \tag{131}$$

The most convenient representation is

$$\Im_{\rm int} = \frac{1}{2\alpha}C(2) + \frac{1}{2\beta}C(1)^2 + \frac{1}{2\mu}\|S\|^2 \tag{132}$$

where $\|S\|$ is the scalar magnitude of spin,

$$\|S\|^2 = -\frac{1}{2}\,{\rm Tr}(S^2) \tag{133}$$

the meaning of constants is as previously. Again it is convenient to use the incompressibility-dilatations splitting

$$\Im_{\rm int} = \frac{1}{2(I+A)}C_{SL(n)}(2) + \frac{1}{2n(I+A+nB)}p^2 + \frac{I}{2(I^2-A^2)}\|S\|^2. \tag{134}$$

Just as previously, for stabilizing dilatations we have to use Hamiltonians with extra introduced dilatational potentials

$$H = \Im_{\rm int} + \mathcal{V}(q). \tag{135}$$

And, similarly, as in affine-metrical models, we conclude that on the purely geodetic incompressible level there exists an open family of bounded and an open family of non-bounded solutions. The same is true for compressible models with appropriately chosen dilatation-stabilizing terms. And even for more general non-geodetic models with appropriate potentials depending only on deformation invariants, $\mathcal{V}(K_1,\ldots,K_n)$. But, of course, particularly interesting are models geodetic in the incompressible factor of $LI(U,V)$.

It is instructive to discuss the relative equilibria in our models. In doubly-invariant affine-affine Casimir models the general solution was given by the family of one-parameter groups and their cosets. It is exactly so in mechanics of spherically symmetric rigid body and other mechanical models with doubly-invariant kinetic energies on Lie groups as configuration spaces. But it is well known that it is no longer the case in mechanics of anisotropic rigid body. There exist there solutions of the type: one-parameter groups and their cosets, but they are very special solutions, relatively small subset within the general solution. Nevertheless, solutions of this kind are very interesting, because they enable one to get some understanding of the general solution structure, at least on the qualitative level. Solutions of this kind are an important special case of what was called by Marsden, Ratiu and others "relative equilibria" [1, 2, 18, 8]. The study of such relative equilibria and their stability is an important subject in the theory of mechanical systems based on Lie groups. Let us consider the problem of relative equilibria in the afore-mentioned geodetic models of affinely-rigid body.

We begin with affine-metrical geodetic models, i.e. ones invariant under $GL(V) \times SO(U, \eta)$. One can show, there exist solutions of the form

$$\varphi(t) = \varphi_0 \mathrm{e}^{Ft} \tag{136}$$

where $\varphi_0 \in LI(U, V)$ is arbitrary, $F \in L(U) \simeq \mathfrak{gl}(U)$ is η-normal, i.e. $[F, F^{\eta T}] = 0$ and $(F^{\eta T})^A_B := \eta^{AC}\eta_{BD}F^D_C$, thus, F does commute with its η-transpose. In particular, F may be η-symmetric or η-antisymmetric. Analogously for incompressible models: $\varphi_0 \in SLI(U, V)$ (or analytically: $\varphi_0 \in SL(n, \mathbb{R})$) is an arbitrary volume-preserving configuration, and $F \in SL(U)'$ is an arbitrary trace-less and η-normal linear mapping of U into itself.

Stationary solutions, i.e. relative equilibria for the metrical-affine geodetic models (i.e. ones invariant under $SO(V, g) \times GL(U)$) are built in a sense symmetrically with respect to the previous ones. Namely, they have the form:

$$\varphi(t) = \mathrm{e}^{Et}\varphi_0 \tag{137}$$

where $\varphi \in LI(U, V)$ is arbitrary, $E \in L(V) \simeq \mathfrak{gl}(V)$ is g-normal, $[E, E^{gT}] = 0$, and $(E^{gT})^i_j := g^{ik}g_{jl}E^l_k$. In particular this holds, when E is g-symmetric, or g-antisymmetric. And again analogously for incompressible models; $\varphi_0 \in SLI(U, V)$ (analytically $\varphi_0 \in SL(n, \mathbb{R})$) is arbitrary and $E \in \mathfrak{sl}(V)$ is an arbitrary trace-less g-normal linear mapping of V into V.

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