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SYMMETRY APPROACH AND EXACT SOLUTIONS IN HYDRODYNAMICS

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Abstract. The application of symmetry analysis in hydrodynamics is illustrated by two examples. First is a description of all irrotational barochronous motions of ideal gas. The second is an exact solution of magnetohydrodynamics equations for infinitely conducting media, which describes the flow of so called "special vortex" type.

1. Introduction

The group-theoretical method is proved to be one of the most powerful tool for the construction of exact solutions for various nonlinear differential equations [4, 5]. The method is based on the continuous symmetries of the investigated equations. The complete set of the continuous transformations, which preserve the equations, generates its Lie group of symmetries. Each subgroup of the symmetry group gives the source of an exact solution or a symmetry reduction for the equations. The systematic use of group analysis method to study concrete models of mathematical physics consists of the following three steps. These are: calculation of symmetry group, construction of its optimal system of subgroups and obtaining of classes of both invariant and partially invariant solutions. Realization of all these steps is algorithmic and approved for the wide set of mathematical models by many authors.

In the present work we observe two particular examples of exact solutions for Euler equations of ideal compressible fluid and for ideal magnetohydrodynamics equations (MHD). First we describe all irrotational ideal gas motions, which are simultaneously **barochronous**, i.e., $\operatorname{rot}\mathbf{u} = \mathbf{0}$ and $p = p(t)$ (pressure depends only on time). This class of solutions is a partially invariant from group-theoretical point of view. The Chupakhin's results on investigation of barochronous gas motions allow to reduce the stated problem to the following: how to describe all real-valued functions of three arguments, which have a Hesse matrix with constant algebraic invariants. The latter problem is completely solved with the help of equivalence transformations of investigated system. It is shown that under irrotational barochronous gas motions the dependence of velocity vector on spatial coordinates is linear and of special kind. The explicit formulas for general solution are given.

The second example is a solution of ideal MHD, which is partially invariant with respect to the group of rotations $\mathrm{O}(3)$. The classical rotationally invariant solution of the system of differential equations is the solution where all sought functions depend on radial coordinate only. From group-theoretical point of view, rotationally invariant solution is a singular $\mathrm{O}(3)$-invariant solution. The nonsingular $\mathrm{O}(3)$-invariant solution do not exist since the set of its invariants does not cover all sought functions (only two of three components of fluid's velocity vector field can be derived from the invariants). However, one can sought for $\mathrm{O}(3)$-partially invariant solution. First, this type of solutions was successfully investigated for Euler equations by Ovsiannikov [7]. He have obtained an overdetermined reduced system of equations, found all its compatibility conditions and describes the main properties of fluid flows, governed by the solution. In compliance with the title of Ovsiannikov's article the solutions of this type are referred as “singular vortex” or “Ovsiannikov's vortex”.

2. Irrotational Barochronous Fluid Flows

The Euler equations for ideal compressible fluid are the following

$$\begin{gathered} D\mathbf{u} + \rho^{-1}\nabla p = 0, \qquad D\rho + \rho\,\mathrm{div}\,\mathbf{u} = 0, \qquad DS = 0 \\ D = \partial_t + u\partial_x + v\partial_y + w\partial_z. \end{gathered} \tag{1}$$

Here $\mathbf{u}$ is the velocity vector, p is the pressure, ρ is the density, and S is the entropy. System (1) is closed by the state equation $p = F(\rho, S)$. We sought for solutions of system (1), which are barochronous

$$p = p(t), \qquad \rho = \rho(t) \tag{2}$$

and irrotational

$$\mathbf{u} = \nabla\varphi \quad \text{for some potential} \quad \varphi(t, x, y, z). \tag{3}$$

According to the state equation $p = F(\rho, S)$ entropy S is a constant $S = \mathrm{const}$.

The group nature of barochronous solutions is the following. Equations (1) admit the Galilean group, which in particular includes transformations of the translations and galilean translations along the spatial axis. Invariants of these transformations are p, ρ and the time t. According to the algorithms of symmetry analysis of differential equations, the functional relations (2) between the invariants single out a partially invariant solution with respect to the group. The complete theoretical investigations of barochronous motions of ideal gas can be found in Chupakhin's works [1, 2]. Our purpose is to add a demand (3) of potentiality of the flow.

2.1. Barochronous Gas Motions

We adduce here some of Chupakhin's results, which will be used further. According to (1) and (2) the barochronous motions of ideal gas are described by the following system of equations

$$D\,\mathbf{u} = 0, \qquad \operatorname{div}\mathbf{u} = -\frac{\rho'(t)}{\rho}, \qquad D = \partial_t + \mathbf{u}\cdot\nabla. \tag{4}$$

The right hand side of the last equation (4) is a function of t only. Thus the system (4) is an overdetermined system for velocity vector $\mathbf{u}$. Description of its compatibility conditions is given in terms of algebraic invariants j_1, j_2, j_3 of Jacobi matrix $J = \partial\mathbf{u}/\partial\mathbf{x}$. Further $j_1 = \operatorname{tr} J, \ldots, j_3 = \det J$.

Theorem 1. *The initial velocity field of barochronous motion has a Jacobi matrix $J_0 = \partial\mathbf{u}_0/\partial\mathbf{x}_0$ with constant algebraic invariants. On the contrary, any stationary vector field with constant algebraic invariants of its Jacobi matrix serves as initial velocity field for some barochronous gas motion.*

It is possible to observe a Cauchy problem for the system (1), (2) with initial data at $t = 0$

$$\mathbf{u}(0,\mathbf{x}) = \mathbf{u}_0(\mathbf{x}), \qquad \rho(0,\mathbf{x}) = \rho_0, \qquad S(0,\mathbf{x}) = S_0. \tag{5}$$

Theorem 2. *Solution of the Cauchy problem* (5) *for the system* (1), (2) *is given by the implicit formulas*

$$\begin{gathered}\mathbf{u} = \mathbf{u}_0(\boldsymbol{\xi}), \qquad \rho = \rho_0/Q, \qquad S = S_0, \qquad \boldsymbol{\xi} = \mathbf{x} - t\mathbf{u} \\ Q = 1 + j_{10}t + j_{20}t^2 + j_{30}t^3.\end{gathered} \tag{6}$$

Here j_{i0} are the initial values of the invariants j_i. From the above it follows that any barochronous solution of gas dynamics equations is completely characterized by its initial velocity field.

Barochronous motions of ideal gas have many interesting properties. The trajectories of particles in such motions are straight lines. However, the whole motion is non-trivial. The typical feature of barochronous motions is the collapse of density at a finite moment of time. At that time all gas particles simultaneously come to some manifold of lower dimension in comparison with the dimension of motion. The behaviour of sonic and contact characteristics of gas dynamics equations in the neighbourhood of collapse is already known [2].

2.2. Irrotational Gas Motions

The irrotational (potential) motions of gas are much more classical object of investigation. Irrotational gas motions are distinguished by a special kind of the velocity field (3). The state equation in the isentropic case $S = \text{const}$ reads $p = f(\rho)$. Integration of the momentum equations in (1) gives the Cauchy–Lagrange integral

$$\varphi_t + \frac{1}{2}|\nabla\varphi|^2 + i(p) = 0. \tag{7}$$

Here $i(p) = \int \rho^{-1}\, \mathrm{d}p$ is the specific enthalpy. The continuity equation provides

$$i_t + \nabla\varphi \cdot \nabla i + a^2 \Delta\varphi = 0. \tag{8}$$

Here $a^2 = f'(\rho)$ is a square of sound speed. Equations (7) and (8) serve for description of irrotational gas motions. One can obtain a single second order equation for φ by substitution of the specific enthalpy i from (7) into (8).

2.3. Irrotational Barochronous Gas Motions

In present paper we combine two properties described above. We look for solutions which are simultaneously irrotational and barochronous. There are two ways of solving the stated problem. First is to start from equations (7) and (8) and to demand the solution to be barochronous. Equations (2) implies that all thermodynamical functions depend only on time: $i = i(t)$, $a = a(t)$. Hence we obtain an overdetermined system of two equations (7) and (8) for one function $\varphi(t, x, y, z)$. This system must be completed to involution.

The second way is to start from barochronous solution taking into consideration property (3). As noted above the description of barochronous gas motions is reduced to the investigation of the equations for the initial velocity field (hereafter we omit zeroes at $\mathbf{u}_0$ and replace $\boldsymbol{\xi}$ by $\mathbf{x}$)

$$\begin{gathered}
u_x + v_y + w_z = j_1 \\
\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} + \begin{vmatrix} v_y & v_z \\ w_y & w_z \end{vmatrix} + \begin{vmatrix} w_z & w_x \\ u_z & u_x \end{vmatrix} = j_2 \\
\begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix} = j_3.
\end{gathered} \tag{9}$$

The constants j_1, j_2, j_3 can take arbitrary real values. Since the equation (3) is valid during the whole time of motion it is also valid for the initial time $t = 0$. Thus, to describe the potential barochronous gas motions it is necessary to study system (9) with the substitution of the velocity field (3). In other words, the investigation of the potential barochronous gas motions is reduced to the description of all functions $\varphi(x, y, z)$, which have a Hesse matrix with constant algebraic invariants. Below we use this second approach. The complete investigation of this problem can be found in [3]. Below we demonstrate only the basic steps of investigation.

2.4. Equivalence Transformations

To simplify the analysis of system (9) it is convenient to transform the vector $\mathbf{j} = (j_1, j_2, j_3)$ to some canonical form. This is performed with the aid of equivalence transformations, which preserve the structure of the system (9) acting only on the constants j_i. The group of equivalence transformations for the system (9) is known [6]. It is generated by the operators

$$\begin{aligned} T_1 &= x^k \partial_{u^k} + 3\partial_{j_1} + 2j_1\partial_{j_2} + j_2\partial_{j_3} \\ T_2 &= u^k \partial_{u^k} - x^k \partial_{x^k} + 2j_1\partial_{j_1} + 4j_2\partial_{j_2} + 6j_3\partial_{j_3} \\ T_3 &= u^k \partial_{x^k} + (2j_2 - j_1^2)\partial_{j_1} + (3j_3 - j_1 j_2)\partial_{j_2} - j_1 j_3 \partial_{j_3}. \end{aligned} \tag{10}$$

It is known [6] that due to transformations T_i any system (9) is equivalent to one of the four canonical systems with vector $\mathbf{j}$ of the form

$$1^0\ (0,0,0); \qquad 2^0\ (1,0,0); \qquad 3^0\ (0,1,0); \qquad 4^0\ (0,-1,0). \tag{11}$$

The principal moment is that in all the four cases $j_3 = 0$. It means that there exists a perfect relationship between functions u, v and w for system (9) in a canonical form. Thus, the initial velocity field of barochronous motion is equivalent to some double $u = u(v, w)$ or "sesquialteral" $u = u(v)$, $w = w(x, y, z)$ wave.

2.5. Double Wave

Let the derivatives of the function φ be related by the expression

$$\varphi_x = u(\varphi_y, \varphi_z). \tag{12}$$

After the finding of absolute integral of equation (12) its general solution is represented in a parametric form

$$\varphi = u(p_1, p_2)x + p_1 y + p_2 z + \psi^0(p_1, p_2). \tag{13}$$

Here $\psi^0(p_1, p_2)$ is the arbitrary function, p_1 and p_2 are parameters, which are related to initial variables by the equalities

$$0 = \frac{\partial u}{\partial p_1}x + y + \frac{\partial \psi^0}{\partial p_1}, \qquad 0 = \frac{\partial u}{\partial p_2}x + z + \frac{\partial \psi^0}{\partial p_2}. \tag{14}$$

Next, it is convenient to move from independent variables (x, y, z) to independent variables (x, p_1, p_2) according to formulas (14). The Jacobian of such transformation

$$\begin{aligned}\Delta &= \frac{\partial(x,y,z)}{\partial(x,p_1,p_2)} \\ &= \left(\frac{\partial^2 u}{\partial p_1^2}x + \frac{\partial^2 \psi^0}{\partial p_1^2}\right)\left(\frac{\partial^2 u}{\partial p_2^2}x + \frac{\partial^2 \psi^0}{\partial p_2^2}\right) - \left(\frac{\partial^2 u}{\partial p_1 \partial p_2}x + \frac{\partial^2 \psi^0}{\partial p_1 \partial p_2}\right)^2\end{aligned} \tag{15}$$

differs from zero due to the arbitrariness in the choice of the function $\psi^0(p_1, p_2)$.

The substitution into the first two equations (9) gives

$$\begin{gathered}-\left(1+\left(\frac{\partial u}{\partial p_2}\right)^2\right)\Delta_1 + 2\frac{\partial u}{\partial p_1}\frac{\partial u}{\partial p_2}\Delta_2 - \left(1+\left(\frac{\partial u}{\partial p_1}\right)^2\right)\Delta_3 = j_1\Delta \\ 1+\left(\frac{\partial u}{\partial p_1}\right)^2 + \left(\frac{\partial u}{\partial p_2}\right)^2 = j_2\Delta.\end{gathered} \tag{16}$$

The notation above means

$$\Delta_1 = x\frac{\partial^2 u}{\partial p_1^2} + \frac{\partial^2 \psi^0}{\partial p_1^2},\ \Delta_2 = x\frac{\partial^2 u}{\partial p_1 \partial p_2} + \frac{\partial^2 \psi^0}{\partial p_1 \partial p_2},\ \Delta_3 = x\frac{\partial^2 u}{\partial p_2^2} + \frac{\partial^2 \psi^0}{\partial p_2^2}. \tag{17}$$

From the last equation in (16) it follows that $j_2 \neq 0$, i.e., the cases 1^0 and 2^0 from classification (11) are not realized here.

Let us consider the remaining cases 3^0 and 4^0. Here $j_1 = 0$, $j_2 = \pm 1$. After the splitting of equations (16) on independent variable x we obtain five equations. Among them the following ones are interesting. Linear with respect to x term in the first equation in (16) gives the equation of minimal surfaces

$$\left(1+\left(\frac{\partial u}{\partial p_2}\right)^2\right)\frac{\partial^2 u}{\partial p_1^2} - 2\frac{\partial u}{\partial p_1}\frac{\partial u}{\partial p_2}\frac{\partial^2 u}{\partial p_1 p_2} + \left(1+\left(\frac{\partial u}{\partial p_1}\right)^2\right)\frac{\partial^2 u}{\partial p_2^2} = 0. \tag{18}$$

The coefficient of x^2 in the second equation in (16) gives the Monge–Ampère equation

$$\frac{\partial^2 u}{\partial p_1^2}\frac{\partial^2 u}{\partial p_2^2} = \left(\frac{\partial^2 u}{\partial p_1 p_2}\right)^2. \tag{19}$$

Thus, surfaces $z = u(x, y)$ with function u satisfying equations (18), (19) are enveloping and minimal. The set of such surfaces turns out to be exhausted by the planes.

Returning to the initial notation we obtain linear relation between the derivatives of function φ

$$\varphi_x = a\varphi_y + b\varphi_z + c, \qquad a, b, c = \text{const}\,. \tag{20}$$

The equation (20) is integrated in the form $\varphi = cx + \varphi^0(ax + y, bx + z)$. Accurate to the transformations of rotation and Galilean translation we can assume that $a = b = c = 0$. Thus, the function φ turns out to be of the following type: $\varphi = \varphi(y, z)$. The similar analysis of two-dimensional case gives that all nonequivalent functions $\varphi(x, y)$, which satisfy the equations (9), are

$$\varphi = \text{const}, \qquad \varphi = \frac{1}{2}x^2, \qquad \varphi = \frac{1}{2}(x^2 - y^2). \tag{21}$$

All "sesquialteral" waves are also reduced to the two-dimensional case.

2.6. Final Result

Summing up all the calculations we can formulate the following statements.

Theorem 3. *The initial velocity field of irrotational barochronous gas motions is equivalent to one of the following:*

a) *constant;*

b) $u = x$, $v = w = 0$

c) $u = x$, $v = -y$, $w = 0$.

Knowledge of initial velocity field of barochronous motion of gas allows one to reconstruct a solution at arbitrary moment of time according to formulas (6). The value of density is regenerated then by the integration of the equation of continuity. Gas pressure is determined from the equation of state. Application of such procedure to the obtained initial field allows us to formulate the following statement.

Theorem 4. *The irrotational barochronous gas motions accurate to unessential constants are described by the following formulas*

$$\begin{gathered} u = \frac{ax}{1 + at}, \qquad v = \frac{by}{1 + bt}, \qquad w = \frac{cz}{1 + ct} \\ \rho = \frac{\rho_0}{(1 + at)(1 + bt)(1 + ct)}, \qquad a,\ b,\ c,\ \rho_0 = \text{const}\,. \end{gathered} \tag{22}$$

3. Singular Vortex In Magnetohydrodynamics

In this part we observe the ideal magnetohydrodynamics equations

$$\begin{aligned} &D\,\rho + \rho\,\mathrm{div}\,\mathbf{u} = 0 \\ &D\,\mathbf{u} + \rho^{-1}\nabla p + \rho^{-1}\mathbf{H} \times \mathrm{rot}\,\mathbf{H} = 0 \\ &D\,p + A(p, \rho)\,\mathrm{div}\,\mathbf{u} = 0 \\ &D\,\mathbf{H} + \mathbf{H}\,\mathrm{div}\,\mathbf{u} - (\mathbf{H} \cdot \nabla)\mathbf{u} = 0 \\ &\mathrm{div}\,\mathbf{H} = 0, \qquad D = \partial_t + \mathbf{u} \cdot \nabla. \end{aligned} \tag{23}$$

Here $\mathbf{u} = (u, v, w)$ is the velocity vector, p and ρ are the pressure and the density, and $\mathbf{H} = (H^1, H^2, H^3)$ is the magnetic field (electric conductivity is infinite). All these functions depend on the time t and the coordinates (x, y, z). The function $A(p, \rho)$ depends the state equation of the fluid.

The equations (23) admit as invariance group the group $\mathrm{O}(3)$ of simultaneous rotations in the spaces $\mathbb{R}^3(\mathrm{x})$, $\mathbb{R}^3(\mathbf{u})$ and $\mathbb{R}^3(\mathbf{H})$. The usual construction of the solution, invariant with respect to $\mathrm{O}(3)$, is a solution, where all sought functions depend only on radial coordinate r and time t and both velocity vector and magnetic field have radial direction. The symmetry analysis of differential equations allows another type of solution, namely, partially invariant one.

It is convenient to observe a spherical coordinate system

$$x = r\sin\theta\cos\varphi, \qquad y = r\sin\theta\sin\varphi, \qquad z = r\cos\theta. \tag{24}$$

Invariants of $\mathrm{O}(3)$ in the space of independent variables and functions are the following

$$t, \quad r, \quad U, \quad M, \quad H, \quad N, \quad \Omega - \Sigma, \quad p, \quad \rho. \tag{25}$$

Here $U = u_r$ and $H = H_r$ are radial components of vector $\mathbf{u}$ and $\mathbf{H}$. Tangential to spheres $r = \mathrm{const}$ components of these vectors are represented as

$$u_\theta = M\cos\Omega, \qquad u_\varphi = M\sin\Omega, \qquad H_\theta = N\cos\Sigma, \qquad H_\varphi = N\sin\Sigma.$$

From (25) is follows that not all sought functions can be determined as the functions of of invariants of $\mathrm{O}(3)$. This fact does not allows to construct an invariant solution with respect to the group $\mathrm{O}(3)$. However, there exist a partially invariant solution with the following representation

$$\begin{aligned}
&U = U(t, r), && M = M(t, r), \qquad H = H(t, r), \qquad N = N(t, r)\\
&\Omega = \omega(t, r, \theta, \varphi), && \Sigma = \sigma(t, r) + \omega(t, r, \theta, \varphi)\\
&p = p(t, r), && \rho = \rho(t, r).
\end{aligned} \tag{26}$$

The presence of one non-invariant function $\omega(t, r, \theta, \varphi)$ makes the solution to be partially invariant with defect $\delta = 1$. The non-invariant function ω is called a “superfluous” function. The classical solution with radial flow and radial magnetic force can be obtained from (26) by taking $M = N = 0$. Further we omit this case as the known one.

Substitution of representation (26) into MHD (23) provides a system Π of nine equations for invariant the functions U, H, N, M, ρ, p and the **superfluous function** ω. This system should be observed as an overdetermined system of first-order PDEs for the function $\omega(t, r, \theta, \varphi)$ under assumption that all invariant functions are known. The compatibility conditions of this system produce the equations for the invariant functions. This procedure is illustrated by the following diagram

$$MHD \xrightarrow{(26)} \boxed{\Pi \text{ -- equations for } \omega} \begin{array}{l} \nearrow \boxed{CS \text{ -- compatible system for } \omega} \\ \searrow \boxed{IS \text{ -- equations for invariant functions}} \end{array} .$$

In order to omit trivial situations, we observe only the case, when function ω is determined with functional arbitrariness. Function ω is determined with only constant arbitrariness if it is possible to express all first-order derivatives of ω from the system Π. To impose a ban on this situation we calculate a matrix of coefficient of ω's derivatives and demand it to be of rank three or less. The demand is satisfied only in the following three cases:

- $M = 0$ – radial velocity field
- $N = 0$ – radial magnetic field
- $\sigma = 0$ – coincidence of derivation angles of the tangential component of the velocity and magnetic vector fields.

All these three cases signify that velocity vector $\mathbf{u}$ and magnetic field vector $\mathbf{H}$ in each point must be coplanar to the radius-vector of the point. Further we observe the most general case $\sigma = 0$, i.e. $\Sigma = \Omega = \omega(t, r, \theta, \varphi)$. It is convenient to introduce

$$M_1 = r^{-1}M, \qquad H_1 = r^2 H, \qquad N_1 = rN, \qquad H_1 = \cos^{-1}\tau. \tag{27}$$

The invariant subsystem IS is reduced to the following

$$\begin{aligned}
&D_0 M_1 + \frac{2}{r}UM_1 - \frac{1}{r^4\rho\cos\tau}N_{1r} = 0, && D_0 = \partial_t + U\partial_r\\
&D_0 N_1 + N_1 U_r - \frac{1}{\cos\tau}M_{1r} - M_1 N_1 \tan\tau = 0 &&\\
&D_0 p + A(p,\rho)\left(U_r + \frac{2}{r}U - M_1\tan\tau\right) = 0 &&\\
&D_0 U + \frac{1}{\rho}p_r + \frac{N_1 N_{1r}}{r^2\rho} - rM_1^2 = 0, && \tau_r = N_1\cos\tau\\
&D_0\rho + \rho\left(U_r + \frac{2}{r}U - M_1\tan\tau\right) = 0, && D_0\tau = M_1.
\end{aligned} \tag{28}$$

This overdetermined system of seven equations for six functions is in involution (compatible and locally solvable) since the compatibility condition of last two equations of (28) (equations for τ) coincide with the second equation in (28).

Equations for the superfluous function ω are

$$\begin{aligned} & N_1\omega_t + (N_1U - H_1M_1)\omega_r = 0 \\ & H_1 \cos\omega\omega_r + N_1\omega_\theta - \tan\tau N_1 \sin\omega = 0 \\ & \sin\theta\sin\omega\omega_\theta - \cos\omega\omega_\varphi - \tan\tau\sin\theta - \cos\theta\cos\omega = 0. \end{aligned} \tag{29}$$

The latter system is also in involution on the solutions of equations (28). The arbitrariness in the general solution of (29) is one function of one argument. The general solution of (29) can be implicitly represented as

$$F(\eta,\zeta) = 0 \tag{30}$$

where F is an arbitrary function of the invariants η and ζ, which are

$$\begin{aligned} & \eta = \sin\theta\cos\omega\cos\tau - \cos\theta\sin\tau \\ & \zeta = \varphi + \arctan\frac{\sin\omega\cos\tau}{\cos\theta\cos\omega\cos\tau + \sin\theta\sin\tau}. \end{aligned}$$

The following question arises: Is it possible to choose a function F (determined by the equation (30)) which is continuous as a function of $\omega(t,r,\theta,\varphi)$ on each sphere $r = \text{const}$? The answer of this question is not known to the author yet.

One can prove that a trajectories of particles and magnetic force lines on this flow are flat curves. However the position and orientation of the plane depends of initial particle's position. Function $\tau(t,r)$ determines a polar angle of particle in the plane of it's motion. In the stationary case ($\partial/\partial_t = 0$) velocity and magnetic field vectors are collinear, therefore the streamlines coincide with magnetic lines. The deeper investigation of the flow requires more specific information on the solution.

3.1. Symmetry of Invariant Subsystem

The determining of the solution of the form (26) is reduced to the investigation of the system (28). The latter serves as an individual object of symmetry analysis. Calculation of admitted group of system (28) (for simplicity $A(p,\rho) = \gamma p$, where γ is polytropic exponent) gives, that admissible Lie group of point transformations is 3-dimensional and its Lie algebra L_3 is generated by the operators

$$\begin{aligned} & X_1 = \partial_t \\ & X_2 = t\partial_t - U\partial_U - M_1\partial_{M_1} + 2\rho\partial_\rho \\ & X_3 = r\partial_r + U\partial_U - N_1\partial_{N_1} - 4p\partial_p - 6\rho\partial_\rho. \end{aligned} \tag{31}$$

Besides, two involutions are admitted

$$\begin{aligned} & \varepsilon_1 : t \to -t, \quad U \to -U, \quad M_1 \to -M_1 \\ & \varepsilon_2 : r \to -r, \quad U \to -U, \quad N_1 \to -N_1. \end{aligned}$$

The optimal system of subalgebras of the Lie algebra L_3 was constructed. The optimal system of subalgebras is a maximal set of nonconjugated (with respect to the action of inner automorphism) subalgebras of L_3. In the case of the algebra L_3, the optimal system is the following:

dim = 1: $\{X_1\}$, $\{X_1 + X_3\}$, $\{X_2 + \alpha X_3\}$

dim = 2: $\{X_1, X_3\}$, $\{X_2, X_3\}$, $\{X_1, X_2 + \alpha X_3\}$

dim = 3: $\{X_1, X_2, X_3\}$.

Each subalgebra generates some symmetry reduction of the system (28).

3.2. Stationary Solution

We observe an invariant with respect to the group of time translation solution of (28) generated by the Lie algebra $\{X_1\}$. Invariants of the group are r and all functions U, M_1, N_1, τ, p, ρ. The functional relations between the invariants ensure that all functions depend only on r. Equations (28) are reduced to the following system of ODE

$$UM_1' + \frac{2}{r}UM_1 - \frac{N_1'}{r^4\rho\cos\tau} = 0 \tag{32}$$

$$UN_1' + N_1U' - \frac{M_1'}{\cos\tau} - M_1N_1\tan\tau = 0 \tag{33}$$

$$Up' + \gamma p(U' + \frac{2}{r},U - M_1\tan\tau) = 0 \tag{34}$$

$$UU' + \frac{1}{\rho}p' + \frac{N_1N_1'}{r^2\rho} - rM_1^2 = 0 \tag{35}$$

$$U\rho' + \rho(U' + \frac{2}{r}U - M_1\tan\tau) = 0 \tag{36}$$

$$\tau' = N_1\cos\tau, \quad U\tau' = M_1. \tag{37}$$

This system can be reduced to the set of first integrals and one first-order ODE.

3.3. Logarithmic and Self-Similar Solution

Another two one-dimensional subalgebras from the optimal system gives another two reductions of (28) to ODEs. Calculating the invariants of each subalgebras and proposing the functional relations between invariants we obtain the representation of the invariant solution.

The subalgebra $\{X_1 + X_3\}$ gives the following representation of the solution

$$\begin{gathered} U = r(v(\lambda) + 1), \quad M_1 = m(\lambda), \quad N_1 = r^{-1}n(\lambda), \quad \tau = \tau(\lambda) \\ p = r^{-4}P(\lambda), \quad \rho = r^{-6}R(\lambda), \quad S = s(\lambda)r^{6\gamma-4}, \quad \lambda = t - \log r. \end{gathered} \tag{38}$$

Substitution of (38) gives a system of ODEs, which involves only the functions v, m, n, P, R and independent variable λ. Curves $\lambda = \text{const}$ are called the level lines of the solution. In this solution the level lines are a logarithmic spirals in the (t, r) plane. Therefore the corresponding solution is called "logarithmic" one.

The last one-dimensional subalgebra from the optimal system is a $\{X_2 + \alpha X_3\}$ with arbitrary real α. The corresponding transformation is a dilatation, therefore the invariant solution is a self-similar one. Calculation of invariants allows one to write the representation of the solution

$$\begin{aligned} &U = t^{\alpha-1}(v(\lambda) + \alpha\lambda), \quad M_1 = m(\lambda)t^{-1}, \quad N_1 = n(\lambda)t^{-\alpha}, \quad \tau = \tau(\lambda) \\ &p = P(\lambda)t^{-4\alpha}, \quad \rho = R(\lambda)t^{2-6\alpha}, \quad S = s(\lambda)t^{6\alpha\gamma-2\gamma-4\alpha}, \quad \lambda = rt^{-\alpha}. \end{aligned} \tag{39}$$

Reduced system of ODEs can be obtained by substitution of (39) into (28). Both systems have a number of first integrals for special values of parameters.

The two-dimensional subalgebras from the optimal system generate so-called "simple" solutions. The number of functionally independent invariants of the corresponding two-dimensional Lie groups in the space $\mathbb{R}^6(U, M_1, N_1, \tau, p, \rho)$ is 6. Besides, there are no invariants, which depends only of t and r. Therefore the representation of solution is constructed by equating all invariants to the constants. Substitution to the initial system gives a finite relation between this constants. Calculation of these relations for all subalgebras shows that all such solutions are reduced to the trivial one. The whole algebra L_3 generates a partially invariant solution of (28), which was not observed yet.

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