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ON UNCERTAINTY, BRAIDING AND JACOBI FIELDS IN GEOMETRIC QUANTUM MECHANICS

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Abstract. Acting within the framework of the geometric quantum mechanics, we give a Jacobi field interpretation of the quantum mechanical uncertainty. Then a link with elliptic curves via the classical integrability of Schrödinger dynamics and the cross ratio interpretation of quantum probabilities is established. Furthermore a geometrical construction of all special unitary representations of the three-strand braid group on the quantum one-qubit space is provided.

1. Integrability in Quantum Mechanics

1.1. Geometry of Quantum Mechanics

We start this paper by briefly reviewing geometric quantum mechanics and, in particular, the completely integrable structure of Schrödinger dynamics in finite dimensional quantum space (cf. [1–4, 6, 15]).

Throughout the paper we assume $\hbar = 1$. Let V be a complex Hilbert space of dimension $n+1$ with a scalar product $\langle\cdot|\cdot\rangle$ which is linear in the second variable. We shall freely use the Dirac notation. The space of pure states in quantum mechanics is the projective space associated to V, denoted by $\mathbb{P}(V)$, of complex dimension n, whose points are the rays $[v]$ (directions) pertaining to nonzero vectors $|v\rangle$. Considering the actions of the unitary group $\mathrm{U}(V)$ associated to $(V, \langle\cdot|\cdot\rangle)$ and its Lie algebra $\mathfrak{u}(V)$, consisting of all skew-hermitian endomorphisms of V (the quantum observables, and with a slight abuse of language), the projective space $\mathbb{P}(V)$ becomes a $\mathrm{U}(V)$-homogeneous Kähler manifold. Furthermore we can identify a point in $\mathbb{P}(V)$ with the projection operator

$$[v] = \frac{|v\rangle\langle v|}{\|v\|^2}$$