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CHEN-SOURIAU CALCULUS FOR ROUGH LOOPS

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Abstract. We study a diffeological Calculus for rough loop spaces.

1. Introduction

Let us consider a topological space N. Let us recall what is a diffeology (see [5,24]). It is constituted of a set of maps ϕ of any open subset U of any $\mathbb{R}^n$ into N. These maps are called plots. They have to satisfy to the following requirements:

- The constant map is a plot.
- If (U, ϕ) and $(U', \phi')U \subseteq \mathbb{R}^n$, $U' \subseteq \mathbb{R}^n$ are two plots such that $U \cap U' = \emptyset$, then $(U \cup U', \phi \cup \phi')$ is still a plot.
- If $j : U \to U'$ is a smooth map, and (U', ϕ') is a plot, $(U, \phi' \circ j)$ is still a plot.

Let us consider as topological space the Hölder based loop space $L_{1/2-\epsilon,x}(M)$ of $1/2 - \epsilon$ Hölder maps γ from S^1 into a compact Riemannian manifold M such that $\gamma(0) = x$. $L_{1/2-\epsilon,x}(M)$ can be endowed with the Brownian bridge measure as well as the heat kernel measure (see [1]). Over it natural functionals are stochastic integrals (see [15,16,18] for the definition of stochastic integrals for the heat kernel measure).

Inspired by the considerations of Chen-Souriau, Léandre has established a differential calculus over $L_{1/2-\epsilon,x}(M)$ which allows to take derivatives of stochastic integrals. Various stochastic cohomology theories were established. The key point is that there are equal to the de Rham cohomology of the Hölder loop space. For the Brownian bridge measure, it is the purpose of [13, 14]. For the heat kernel measure, it is the purpose of [15] in the case where we replace the loop space by a torus group. As a corollary, [14] shows that a stochastic line bundle (with fiber almost surely defined) is isomorphic to a line bundle over the Hölder loop space.

There is a theory, created recently by Lyons [21], which allows to define stochastic integral path by path: this is the theory of rough paths. Brownian loops are almost surely rough loops. Our goal is to define a diffeology over the rough loop space of a Riemannian manifold, and to show that the associated cohomology groups are equal to the cohomology groups of the Hölder loop space. As a corollary, we show that a line bundle over the rough loop space (a subset of the Hölder loop space) is isomorphic to a bundle of the Hölder loop space. We consider the example of Brylinski of the transgression of a three-form in [3]. The study of this example led Léandre in [9–12] to the introduction of line bundle over the loop space whose transition functional contain stochastic integrals in their definition. By using the theory of rough loops, these stochastic integrals are surely defined and continuous over the rough loop space. We refer to [4] for the studying of such an example. The difference with [4] is that we use here an intrinsic definition of the based rough loop space of a manifold.

2. The Topological Space of Rough Loops

Let M be a Riemannian manifold of dimension d. Let $\mathbb{R}^d = T_x(M)$.
Over $\mathbb{R}^d$, we consider the space of rough loops $G^p(\mathbb{R}^d)$ where $p \in (2,3)$. It is the completion for the distance δ_p of flat C^1 loops

$$\delta_p(\gamma,\gamma') = \mathrm{Var}_p^1(\gamma,\gamma') + \mathrm{Var}_p^2(\gamma,\gamma') \tag{1}$$

where

$$\mathrm{Var}_p^1(\gamma,\gamma') = \sup\left(\sum |\gamma(t_i) - \gamma(t_{i+1}) - \gamma'(t_i) + \gamma'(t_{i+1})|^p\right)^{1/p} \tag{2}$$

where we take the supremum over all the subdivision t_i of S^1. $\mathrm{Var}_p^2(\gamma,\gamma')$ is defined as follows: we consider the iterated integral $\gamma_{1,s,t} = \int_{s<u<v<t} \mathrm{d}\gamma_u \otimes \mathrm{d}\gamma_v$. Then

$$\mathrm{Var}_p^2(\gamma,\gamma') = \sup\left(\sum |\gamma_{1,t_i,t_{i+1}} - \gamma'_{1,t_i,t_{i+1}}|^{p/2}\right)^{2/p} \tag{3}$$

where we take the supremum over all the subdivision t_i of the circle.
In the sequel, we will replace the variational norm Var_p^1 by the **Hölder norm**

$$\sup \frac{|\gamma(s) - \gamma(t)|}{|s-t|^{1/p}} = H_p^1(\gamma) \tag{4}$$

which is stronger than the variational norm. We put $1/p = 1/2 - \epsilon$. For the set of rough loops $G^p(T_x(M))$, we will consider the completion of the space of C^1 loops with respect of the distance

$$H_p^1(\gamma,\gamma') + \mathrm{Var}_p^2(\gamma,\gamma') = \delta'_p(\gamma,\gamma'). \tag{5}$$

Lemma 1. *Let F be a smooth function from $S^1 \times T_x(M)$ into $T_x(M)$ such that $F(0,0) = F(1,0) = 0$. Then the Nemystky map Ψ*

$$\gamma \to \{s \to F(s, \gamma(s))\} \tag{6}$$

is a continuous function on the space of rough loops endowed with the distance δ'_p.

Proof: The Nemytsky map is continuous for the Hölder norm (see [14, Theorem A.1]). It remains to study the continuity for the second variational distance. But

$$F(t, \gamma(t)) = \int_0^t \frac{\partial}{\partial t} F(u, \gamma(u))\,\mathrm{d}u + \int_0^t \frac{\partial}{\partial y} F(u, \gamma(u))\,\mathrm{d}\gamma(u). \tag{7}$$

It is classical (see [20,21, Corollary 3.1]) that the integral of a one-form is continuous over the rough loop space. Moreover, $s \to (s, \gamma(s))$ is a rough path. Therefore the result. □

We will define the space of rough loops $G^p_x(M)$ starting from x as a topological manifold modelled on the space of rough loops in $T_x(M)$. The main difficulty to overcome is that the flat rough loop space is **not a linear space**.

Let $s \to \gamma_{sm}(s)$ a smooth loop issued from x. We write

$$\gamma(s) = \exp_{\gamma_{sm}(s)}[\tau_s(\gamma_{sm}) X_s] \tag{8}$$

where $\exp_{\gamma_{sm}(s)}$ is the exponential map in $\gamma_{sm}(s)$ for the Riemannian distance and $s \to \tau_s(\gamma_{sm})$ the parallel transport along the path $s \to \gamma_{sm}(s)$. If γ is closed enough for the uniform distance of γ_{sm}, we will say that γ belongs to $G^p_x(M)$ if $s \to X_s$ belongs to $G^p(T_x(M))$. This notion is consistent by Lemma 1. The map (8) realizes by definition a local homeomorphism between $G^p_x(M)$ and $G^p(T_x(M))$. The local trivialization (8) produce a topology on $G^p_x(M)$, which endows it with the structure of a topological manifold modelled on $G^p(T_x(M))$.

We get

Lemma 2. *Let F be an application from $[0,1] \times M$ into M such that $F(0,0) = F(1,0) = x$. Let Ψ be the application $\gamma \to \{s \to F(s, \gamma(s))\}$. Ψ is a continuous application from $G^p_x(M))$ into $G^p_x(M)$.*

Proof: In a convenient chart, Ψ is given by

$$\gamma \to \{s \to F^1(s, \gamma(s))\} \tag{9}$$

where F^1 is a convenient function with values in $T_x(M)$. The result arises by Lemma 1. □

3. A Diffeology on the Space of Rough Loops

We will define a diffeology on $G_x^p(M)$. Let us consider an open subset U of $\mathbb{R}^n$. We consider a cover of U by open subset U_i of $\mathbb{R}^n$. We consider an element γ_i of $G_x^p(M)$ and a tubular neighborhood O_i in $S^1 \times M$ of the graph of γ_i. Over O_i, we consider a map $\phi_i : U_i \to G_x^p(M)$ $\phi_i(u,s) = F_i(u,s,\gamma_i(s))$ where F_i is smooth from $U_i \times O_i$ into M. We suppose that $\phi_i = \phi_j$ over $U_i \cap U_j$. The collection of (U_i, ϕ_i) constitutes a plot (U, ϕ) from U into $G_x^p(M)$. The fact that $s \to \phi_i(u,s)$ is a rough loop can be seen by the considerations of the previous part.

The set of (U, ϕ) constitutes clearly a diffeology.

We can define what is a n-form relatively to the diffeology on $G_x^p(M)$ (see [5,24]).

Definition 1. *A n-form σ on $G_x^p(M)$ smooth in the Chen-Souriau sense is given by a the assignment of a n-form $\phi^*\sigma$ on U associated to any plot (U,ϕ). The set of $\phi^*\sigma$ has to satisfy to the following requirement: if $j : U_1 \to U_2$ is a smooth map and (U^2, ϕ^2) is a plot, and if $\phi^2 \circ j = \phi^1$ is the composite plot, we have*

$$\phi^{1*}\sigma = j^*\phi^{2*}\sigma. \tag{10}$$

Let us give an example of a form on $G_x^p(M)$. Let ω be a closed Z-valued three-form on M. Let us consider the transgression $\tau(\omega)$ of it (see [3])

$$\tau(\omega) = \int_{S^1} \omega(\mathrm{d}\gamma(s), ., .). \tag{11}$$

It defines a two-form over $G_x^p(M)$ in the Chen-Souriau sense

$$\begin{aligned} &\phi_i^*\tau(\omega)(X,Y) \\ &= \int_{S_1} \omega(F_i(u,s,\gamma_i(s))), \mathrm{d}_s F_i(u,s,\gamma_i(s)), \partial_X F_i(u,s,\gamma_i(s)), \partial_Y F_i(u,s,\gamma_i(s)) \end{aligned} \tag{12}$$

where X and Y are vector fields over U_i.

The fact that it defines a smooth form on $U = \cup U_i$ is proved by the following lemma.

Lemma 3. *Let $\gamma \in G_x^p(M)$. Let σ_u a form which depends smoothly from u in the open subset U of $\mathbb{R}^n$. Then the* ***Stratonovitch integral*** *$\int_{S^1} \langle \sigma_u, \mathrm{d}\gamma(s)\rangle$ depends smoothly on u.*

Proof: For any multi-index α, we can define the Stratonovitch integral, by using Lyons's theory

$$\int_{S^1} \left\langle \frac{\partial^\alpha}{\partial u^{(\alpha)}}\sigma_u, \mathrm{d}\gamma(s) \right\rangle. \tag{13}$$

It is bounded over each compact of U. By Sobolev imbedding theorem, we deduce that $\int_{S^1} \langle \sigma_u, \mathrm{d}\gamma(s)\rangle$ depends smoothly on u. □

To a form σ smooth in the Chen-Souriau sense, we can associated its exterior derivative defined for a plot (U, ϕ) by

$$\phi^* \mathrm{d}\sigma := \mathrm{d}\phi^* \sigma. \tag{14}$$

In particular, $\mathrm{d}\tau(\omega) = 0$. Namely, we can approach in (U_i, ϕ_i) γ_i by smooth loops γ_i^n. $\phi_i^* \tau(\omega)$ is approached by $\phi_i^{n*} \tau(\omega)$ for the smooth topology. But on U_i $\mathrm{d}\phi_i^{n*} \tau(\omega) = 0$ (see [3]).

4. Isomorphism of Cohomology

This part is an adaptation of the proof of Theorem 2.9 of [14]. We will refer to it without to give all the details.

Let us recall that an open subset for the Hölder topology is still an open subset of $G_x^p(M)$.

Let us give some notations. Let x_i be a finite set of elements of M such that the balls $B(x_i; \delta)$ constitute an open cover of M for δ small enough. We consider the set of polygonal curves $\gamma^{i,n}$ associated to a subdivision $t_k = k/n$ of the interval $[0, 1]$ such that $d(\gamma^{i,n}(t_k), \gamma^{i,n}(t_{k+1})) \leq 2\delta$. Moreover, between t_k and t_{k+1}, the polygonal curve is the unique geodesic joining the two points $\gamma^{i,n}(t_k)$ and $\gamma^{i,n}(t_{k+1})$ and each $\gamma^{i,n}(t_k)$ is some of the points x_i. When i and n describe the set of integers, the set of open balls for the uniform distance $B(\gamma^{i,n}, \delta")$ constitutes an open cover $O_{i,n}$ of $L_{1/2-\epsilon,x}(M)$ for $\delta"$ small enough.

Let us recall the following statement (see [14, Theorem 2.4]): Associated to the cover $O_{i,n}$, there exists a smooth partition of unity $g_{i,n}$ for the Hölder topology. Moreover, each $g_{i,n}$ defines a functional smooth in the Chen-Souriau sense over $G_x^p(M)$, because a plot (U, ϕ) is smooth for the Hölder topology.

Let us put $\alpha = (i, n)$. There exists a natural order over the system of multiindices α. In the sequel, $\alpha_1 < \alpha_2 < \cdots < \alpha_n$. O_α is contractible as well as $O_{\alpha_1,\ldots,\alpha_n} = \cap O_{\alpha_i}$.

We can consider plots constrained to belong to $O_{\alpha_1,\ldots,\alpha_n}$ and we get Chen-Souriau cohomology groups associated to $O_{\alpha_1,\ldots,\alpha_n}$.

Lemma 4. *The Chen-Souriau cohomology groups of $O_{\alpha_1,\ldots,\alpha_n}$ supposed nonempty are equal to zero in degree different to zero and to $\mathbb{R}$ in degree 0.*

Proof: We proceed as in [14, p. 127]. There exists a functional from $O_{\alpha_1,\ldots,\alpha_n} \times [0, 1]$ on $O_{\alpha_1,\ldots,\alpha_n}$ such that:

i) $F_{\alpha_1,\ldots,\alpha_n}(\gamma, t)(s) = F_{\alpha_1,\ldots,\alpha_n}(s, \gamma(s), t)$.

ii) $F_{\alpha_1,\ldots,\alpha_n}(., ., .)$ is smooth in s, $\gamma(s)$ and t.

iii) $F_{\alpha_1,\ldots,\alpha_n}(s, \gamma(s), 1) = \gamma(s)$.

iv) $F_{\alpha_1,\dots,\alpha_n}(s,\gamma(s),0) = \gamma_{\alpha_1,\dots,\alpha_n}(s)$ where $\gamma_{\alpha_1,\dots,\alpha_n}(.)$ is any smooth loop belonging to $O_{\alpha_1,\dots,\alpha_n}$.

If (U,ϕ) is a plot with values in $O_{\alpha_1,\dots,\alpha_n}$, we can construct a retraction plot $(U \times [0,1], \phi^{ext})$ which is still a plot with values in $O_{\alpha_1,\dots,\alpha_n}$ by putting

$$\phi^{ext}(u,t) = F_{\alpha_1,\dots,\alpha_n}(\phi(u),t). \tag{15}$$

This retracts the plot ϕ into a constant plot.

By using Cartan's homotopy formula (see [14, (2.16)]), we deduce that a closed form in Chen-Souriau sense over $O_{\alpha_1,\dots,\alpha_n}$ is exact if its degree is not zero and is constant if its degree is 0 (see [14, Lemma 2.7]). Therefore the result. □

By using a spectral sequence as in [2] and comparing de Rham cohomologies with Cech cohomology associated to the cover O_α, we have the analogue of Theorem 2.9 of [14].

Theorem 1. *The cohomology groups in Chen-Souriau sense of $G_x^p(M)$ are equal to the de Rham cohomology groups of $L_{1/2-\epsilon,x}(M)$.*

5. Isomorphism of Line Bundles

Let us recall the definition of a $\mathbb{Z}$-valued n-form in the Chen-Souriau sense on $G_x^p(M)$. Let Δ^n be the canonical n-simplex in $\mathbb{R}^n$. Let us consider a plot (Δ^n,ϕ) with values in $G_x^p(M)$. We can define its oriented boundary. We can add and subtract simplices (Δ^n,ϕ). If the boundary destroy, we say that we are in presence of a n-cycle.

If (Δ^n,ϕ) is a n-simplex and σ a n-form in Chen-Souriau sense over $G_x^p(M)$, we define

$$\int_{\phi\Delta^n} \sigma := \int_{\Delta^n} \phi^*\sigma. \tag{16}$$

This allows to define the integral of a n-form in Chen-Souriau sense over a n-cycle. We say that σ is $\mathbb{Z}$-valued if its integral over any n-cycle is an integer and if σ is closed.

Let us suppose that $\Pi_1(M) = \Pi_2(M) = 0$.

Let us consider a $\mathbb{Z}$-valued two-form on $G_x^p(M)$ called σ_1.

Let us introduce a system of smooth loops γ_α such that the system of balls for the uniform distance $B(\gamma_\alpha,\delta)$ for δ small enough constitute a cover of $L_{1/2-\epsilon,x}(M)$. Let $x(.)$ be the constant loop. If $\gamma \in B(\gamma_\alpha;\delta)$, there is a distinguished l_α joining γ to $x(.)$: we go from γ to γ_α by the curve $l_\alpha(\gamma)(t)(s) = \exp_{\gamma_\alpha(s)}[t(\gamma(s)-\gamma_\alpha(s))]$ where $\exp_{\gamma_\alpha(s)}$ is the Riemannian exponential in $\gamma_\alpha(s)$ and $\gamma(s)-\gamma_\alpha(s)$ is the unique vector field over $\gamma_\alpha(s)$ of the unique geodesic joining $\gamma_\alpha(s)$ to $\gamma(s)$. We continue the path $l_\alpha(\gamma)(t)$ by a path joining γ_α to $x(.)$ which does not depend on

γ. If γ belongs to $B(\gamma_\alpha;\delta) \cap B(\gamma_\beta;\delta) \cap G^p_x(M)$, we can produce a system of transition functionals as follows: we join the two loops $l_\alpha(\gamma)(t)(.)$ and $l_\beta(\gamma)(t)(.)$ in the small triangle constituted of γ, γ_α and γ_β by using exponential charts. We use the fact that $\Pi_2(M) = 0$ in order to find a surface in the smooth loop space whose boundary is the curve joining $x(.)$ to γ_α, the exponential curve joining γ_α to γ_β and the curve joining γ_β to $x(.)$. We produce a surface $S_{\alpha,\beta}(\gamma)$ in $G^p_x(M)$ which is a union of two-simplices. We put

$$\rho^1_{\alpha,\beta}(\gamma) = \exp\left[-2\pi\mathrm{i}\int_{S_{\alpha,\beta}(\gamma)} \sigma_1\right]. \tag{17}$$

From the fact that σ_1 is closed and $\mathbb{Z}$-valued, we deduce for $\gamma \in B(\gamma_{\alpha_1};\delta) \cap B(\gamma_{\alpha_2};\delta) \cap B(\gamma_{\alpha_3};\delta) \cap G^p_x(M)$ that

$$\rho^1_{\alpha_1,\alpha_2}(\gamma)\rho^1_{\alpha_2,\alpha_3}(\gamma)\rho^1_{\alpha_3,\alpha_1}(\gamma) = 1 \tag{18}$$

and that

$$\rho^1_{\alpha,\beta}(\gamma)\rho^1_{\beta,\alpha}(\gamma) = 1. \tag{19}$$

Moreover, the functionals $\rho^1_{\alpha,\beta}$ are smooth in the Chen-Souriau sense. They define a line bundle over $G^p_x(M)$ which is smooth in the Chen-Souriau sense Λ^1.
But

$$\sigma_1 = \mathrm{d}\sigma_2 + \sigma_3 \tag{20}$$

where σ_3 is a true form over $L_{1/2-\epsilon,x}(M)$.
The functional $\rho^3_{\alpha,\beta}(\gamma)$ is defined analogously to $\rho^1_{\alpha,\beta}(\gamma)$, by replacing σ^1 with σ^3. The map $\gamma \to \rho^3_{\alpha,\beta}(\gamma)$ is continuous for the Hölder topology on $G^p_x(M)$. It defines a continuous line bundle Λ^3 for the Hölder topology on $G^p_x(M)$.
But

$$\rho^1_{\alpha,\beta}(\gamma) = \rho^3_{\alpha,\beta}(\gamma)\exp\left[-2\mathrm{i}\pi\int_{S_{\alpha,\beta}(\gamma)} \mathrm{d}\sigma_2\right]. \tag{21}$$

By Stokes theorem

$$\exp\left[-2\mathrm{i}\pi\int_{S_{\alpha,\beta}(\gamma)} \mathrm{d}\sigma_2\right] = \exp\left[-2\mathrm{i}\pi\int_{l_\alpha(\gamma)} \sigma_2\right]\exp\left[2i\pi\int_{l_\beta(\gamma)} \sigma_2\right]. \tag{22}$$

This means that Λ^1 and Λ^3 are isomorphic in Chen-Souriau sense over $G^p_x(M)$.
Let us consider the case of $\tau(\omega)$.
We remark that

$$\int_{S_{\alpha_1,\alpha_2}(\gamma)} \tau(\omega) + \int_{S_{\alpha_2,\alpha_3}(\gamma)} \tau(\omega) + \int_{S_{\alpha_3,\alpha_1}(\gamma)} \tau(\omega) \in \mathbb{Z} \tag{23}$$

over $G^p_x(M)$ (see [3,4]). We can replace σ^1 by $\tau(\omega)$ in (17) and (23) show that the transition functionals $\rho_{\alpha,\beta}(\gamma)$ got by this procedure still satisfy (18) and (19).

Moreover, if we use Lemma 3, we see that $\rho_{\alpha,\beta}(\gamma)$ is continuous on $G_x^p(M)$ for the rough loop topology. We deduce

Theorem 2. *$\tau(\omega)$ determines a continuous line bundle over $G_x^p(M)$ for the rough loop topology.*

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