Ninth International Conference on
Geometry, Integrability and Quantization
June 8–13, 2007, Varna, Bulgaria
Ivaïlo M. Mladenov, Editor
**SOFTEX**, Sofia 2008, pp 265–279

# THE GENERAL NOTION OF A CURVATURE IN CATASTROPHE THEORY TERMS*

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**Abstract.** We introduce a new notion of a curvature of a superconnection, different from the one obtained by a purely algebraic analogy with the curvature of a linear connection. The naturalness of this new notion of a curvature of a superconnection comes from the study of the singularities of smooth sections of vector bundles (Catastrophe Theory). We demonstrate that the classical examples of obstructions to a local equivalence: exterior differential for two-forms, Riemannian tensor, Weil tensor, curvature of a linear connection and Nijenhuis tensor can be treated in terms of some general approach. This approach, applied to the superconnection leads to a new notion of a curvature (proposed in the paper) of a superconnection.

## 1. A Brief Review of the Notion of a Superconnection

The notions of a superconnection and of the corresponding supercurvature were introduced by Quillen in 1985 [7]. In this section we give a brief review of the matter and introduce the basic notations.

By $\xi = (E, p, M)$ we denote a vector bundle over the manifold $M$ ($\dim M = m$, $\dim(\xi) = n$), by $\xi^*$ – the dual bundle and by $C^\infty(\xi)$ – the space of the vector fields, i.e., the space of the smooth sections of the bundle $\xi$. Respectively $\Omega^k(M) = C^\infty(\Lambda^k T^*(M))$ is the the space of the differential $k$-forms on the manifold $M$ and

$$\Omega(M) = \bigoplus_{k=0}^{m} \Omega^k(M) = C^\infty\left(\bigoplus_{k=0}^{m} \Lambda^k T^*(M)\right)$$

*In memoriam of our dear friend and colleague Ventzeslav Rizov.

is the space of the nonhomogeneous forms on $M$ and $\Omega^k(\xi) = C^\infty(\Lambda^k T^*(M) \otimes \xi)$ is the space of the differential $k$-forms with values in $\xi$ and $\Omega(\xi) = \oplus_{k=0}^m \Omega^k(\xi)$ is the space of the nonhomogeneous forms with values in $\xi$. By a linear connection $\nabla$ on $\xi$ we understand a linear differential operator $\nabla : \Omega^0(\xi) \longrightarrow \Omega^1(\xi)$ with the property $\nabla(f.\psi) = \mathrm{d}f \otimes \psi + f\nabla(\psi)$, $\psi \in \Omega^0(\xi)$, $f \in \Omega^0(M)$. The space of the linear connections is an affine space with a linear group $\Omega^1(\xi^* \otimes \xi)$. If we choose an arbitrary linear connection $\nabla_0$ as "an origin" then for every linear connection $\nabla$ on $\xi$ we have

$$\nabla = \nabla_0 + A, \qquad A \in \Omega^1(\xi^* \otimes \xi). \tag{1}$$

The connection $\nabla$ generates a covariant differential $\mathrm{d}^\nabla : \Omega^k(\xi) \longrightarrow \Omega^{k+1}(\xi)$ defined by the property $\mathrm{d}^\nabla(\alpha \otimes \psi) = \mathrm{d}\alpha \otimes \psi + (-1)^k \mathrm{d} \wedge \nabla(\psi)$, $\alpha \in \Omega^k(M)$, $k = 1, 2, \ldots, m$ and $\psi \in \Omega^0(\xi)$. Its square $\mathrm{d}^\nabla \circ \mathrm{d}^\nabla : \Omega^k(\xi) \longrightarrow \Omega^{k+2}(\xi)$ is an $\Omega^0(M)$-linear operator and as a consequence $F^\nabla = \mathrm{d}^\nabla \circ \mathrm{d}^\nabla : \Omega^0(\xi) \longrightarrow \Omega^2(\xi)$ is a differential operator of zero order, i.e., $F^\nabla$ is a tensor, $F^\nabla \in \Omega^2(\xi^* \otimes \xi)$. Let $\{x^\mu\}$ denotes (local) coordinates on $M$, $\{e_a\}$ – a (local) basis in $\xi$ and $\{e^a\}$ – the corresponding dual basis in $\xi^*$. In local coordinates

$$(\nabla(\psi))^a_\mu = \partial_\mu \psi^a + A^a_{\mu b} \psi^b \tag{2}$$

and

$$F^{\nabla\, a}_{\mu\nu\, b} = (\partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu])^a_b. \tag{3}$$

The upper notions have an algebraic analogy in the case of a $\mathbb{Z}_2$ graded super vector bundle [7]. Let $\xi = \xi_+ \oplus \xi_-$ be a $\mathbb{Z}_2$ graded vector bundle over the manifold $M$. Then the dual bundle $\xi^* = \xi^*_+ \oplus \xi^*_-$ is also $\mathbb{Z}_2$ graded in a natural way. The induced $\mathbb{Z}_2$ grading in $\Omega(\xi)$ is given by

$$\Omega(\xi) = \Omega(\xi)_+ \oplus \Omega(\xi_-)$$

where

$$\Omega(\xi)_+ = \bigoplus_k \left( \Omega^{2k}(\xi_+) \oplus \Omega^{2k+1}(\xi_-) \right)$$

and

$$\Omega(\xi)_- = \bigoplus_k \left( \Omega^{2k+1}(\xi_+) \oplus \Omega^{2k}(\xi_-) \right).$$

Let $\nabla$ be a linear connection on the vector bundle $\xi$ compatible with the $\mathbb{Z}_2$ grading of $\xi$,

$$\nabla : \Omega^0(\xi_\pm) \longrightarrow \Omega^1(\xi_\pm)$$

i.e., $\nabla$ is an odd linear operator.

Let $\chi \in \Omega^0(\xi^* \otimes \xi)$ be an odd tensor field, i.e., $\chi(x) : \xi_{x\pm} \longrightarrow \xi_{x\mp}$ (or $\chi(x) \in (\xi^* \otimes \xi)_{x-}$). By definition (proposed by Quillen) a superconnection $\nabla_s$ is the odd linear operator

$$\nabla_s = \nabla + \chi : \Omega^0(\xi)_\pm \longrightarrow \Omega^{0,1}(\xi)_\mp$$

where $\Omega^{0,1}(\xi) = \Omega^0(\xi) \oplus \Omega^1(\xi)$. It is easy to see that the superconnection $\nabla_s$ has the property

$$\nabla_s(f.\psi) = \mathrm{d}f \otimes \psi + f\nabla_s(\psi), \qquad f \in \Omega^0(M), \qquad \psi \in \Omega^1(\xi).$$

In more details, if we write $\psi = \begin{bmatrix}\psi_+\\ \psi_-\end{bmatrix}$, where $\psi_+$ and $\psi_-$ are the even and the odd parts of the super vector field $\psi$, respectively, then

$$\nabla_s \begin{bmatrix}\psi_+\\ \psi_-\end{bmatrix} = \begin{bmatrix}\nabla & \chi_{+-}\\ \chi_{-+} & \nabla\end{bmatrix} \begin{bmatrix}\psi_+\\ \psi_-\end{bmatrix} = \begin{bmatrix}\nabla\psi_+ + \chi_{+-}\psi_-\\ \nabla\psi_- + \chi_{-+}\psi_+\end{bmatrix}.$$

The space of the superconnections is again an affine space and if we choose an arbitrary superconnection $\nabla_{s0}$ as an origin, then for any other superconnection $\nabla_s$ we have

$$\nabla_s = \nabla_{s0} + A_s, \qquad A_s \in \Omega^{0,1}(\xi^* \otimes \xi)_-.$$

The covariant differential

$$\mathrm{d}^{\nabla_s} : \Omega^k(\xi) \longrightarrow \Omega^{k,k+1}(\xi)$$

can be defined by a purely algebraic analogy with the "classical" case

$$\mathrm{d}^{\nabla_s}(\alpha \otimes \psi) = \mathrm{d}\alpha \otimes \psi + (-1)^k \alpha \wedge \nabla_s(\psi), \qquad \alpha \in \Omega^k(M), \qquad \psi \in \Omega^0(\xi).$$

It is easy to see that again $\mathrm{d}^{\nabla_s} \circ \mathrm{d}^{\nabla_s} : \Omega^k(\xi) \longrightarrow \Omega^{k,k+1,k+2}(\xi)$ is an $\Omega^0(M)$ linear operator. $F^{\nabla_s} = \mathrm{d}^{\nabla_s} \circ \mathrm{d}^{\nabla_s} : \Omega^0(\xi) \longrightarrow \Omega^{0,1,2}(\xi)$ is an even differential operator of order zero, i.e., $F^{\nabla_s}$ is a tensor, $F^{\nabla_s} \in \Omega^{0,1,2}(\xi^* \otimes \xi)$

$$F^{\nabla_s} = \chi^2 + \mathrm{d}^{\nabla}(\chi) + F^{\nabla}.$$

The nonhomogeneous tensor $F^{\nabla}$ defined by an algebraic analogy with the "classical" case is, by definition, the supercurvature of the superconnection $\nabla_s$.

In the classical case the connection $\nabla$ is connected with the parallel transport and the curvature is an obstruction to the flatness of the parallel transport. In other words the curvature of a connection is an obstruction to its local equivalence with the flat connection. In the "super" case there are no natural notions of a parallel transport and of a flat superconnection. But we can look for an obstruction to the local equivalence of two superconnections. To motivate the idea we consider in the next section some classical examples of obstructions to a local equivalence of sections of some bundles as particular examples of a general scheme.

## 2. A List of Obstructions to a Local Equivalence

### 2.1. An Obstruction for an Arbitrary Nondegenerate Differential Two-Form to be Diffeomorphic to the Canonical Symplectic Form

Let $M$ be a smooth even-dimensional manifold, $\dim M = m = 2l$ and let $\omega_0 \in \Omega^2(M)$. In local coordinates $\omega_0 = \sum_{\mu=1}^{l} \mathrm{d}x^\mu \wedge \mathrm{d}x^{l+\mu}$. Every diffeomorphism

$\varphi \in \mathrm{Diff}(M)$ has a natural action on $\Omega^2(M)$. For every $\omega \in \Omega^2(M)$ we have

$$\varphi^*(\omega)_{\mu\nu}(x) = \frac{\partial \varphi^\alpha}{\partial x^\mu}(x)\frac{\partial \varphi^\beta}{\partial x^\nu}(x)\omega_{\alpha\beta}(\varphi(x)).$$

Does there exist (at least locally) $\varphi \in \mathrm{Diff}(M)$ such that $\varphi^*(\omega) = \omega_0$ in the case $\det\{\omega_{\mu\nu}(x)\} \neq 0$? An obstruction to this is the condition $\mathrm{d}\omega \neq 0$. If $\mathrm{d}\omega = 0$ then there exists (at least locally) $\varphi \in \mathrm{Diff}(M)$ such that $\varphi^*(\omega) = \omega_0$. The three-form $\mathrm{d}\omega$ is in some sense the curvature of the two-form $\omega$.

### 2.2. An Obstruction for an Euclidean Metric to be in a Canonical Form

Let $M$ be a smooth manifold, $g_0 = \sum_{\mu=1}^m \mathrm{d}x^\mu \otimes \mathrm{d}x^\mu$ – an Euclidean metric on $M$ in a canonical form and $g$ – an arbitrary Euclidean metric on $M$. Does there exist (at least locally) $\varphi \in \mathrm{Diff}(M)$ such that

$$\varphi^*(g)_{\mu\nu}(x) = \frac{\partial \varphi^\alpha}{\partial x^\mu}(x)\frac{\partial \varphi^\beta}{\partial x^\nu}(x)g_{\alpha\beta}(\varphi(x)) = g_{0\mu\nu}(x) = \delta_{\mu\nu}?$$

The Riemannian tensor $R(g) \neq 0$ is an obstruction to this. If $R(g) = 0$ then there exists (at least locally) a $\varphi \in \mathrm{Diff}(M)$ such that $\varphi^*(g) = g_0$. The Riemannian tensor $R(g)$ is the curvature of the metric $g$.

### 2.3. An Obstruction for an Euclidean Metric to be Conformaly Equivalent to the Canonical Metric

Let $g_0$ and $g$ be the same as in subsection 2.2. Does there exist (at least locally) $\varphi \in \mathrm{Diff}(M)$ such that $\varphi^*(g)(x) = f(x)g_0(x)$, $f(x) \neq 0$? The Weil tensor

$$W(g) = R(g) - \frac{1}{m-2}\left(\mathrm{Ric}(g) - \frac{1}{2(m-2)}r(g).g\right) \wedge g \neq 0$$

where $\mathrm{Ric}(g)$ is the Ricci tensor, $r(g)$ – the scalar curvature, is an obstruction to this [2]. If $W(g) = 0$ then there exists (at least locally) $\varphi \in \mathrm{Diff}(M)$ such that $\varphi^*(g)(x) = f(x)g_0(x)$, $f(x) \neq 0$. The Weil tensor $W(g)$ is the conformal curvature of the metric $g$.

### 2.4. An Obstruction for a Given Connection to be Gauge Equivalent to the Flat One

Let $\xi = (E, p, M)$ be a vector bundle over the manifold $M$ and $\nabla$ be a linear connection on $\xi$ (see (1)). In local coordinates we have

$$\nabla_\mu = \partial_\mu + A_\mu.$$

Let $\varphi \in \mathrm{Aut}^V(\xi)$ be a vertical automorphism of $\xi$. The automorphism $\varphi$ has a natural action on the space of the linear connections. In local coordinates

$$\varphi^*(\nabla)_\mu = \partial_\mu + \varphi^*(A)_\mu, \quad \varphi^*(A)_\mu(x) = \varphi^{-1}(x).A_\mu(x).\varphi(x) + \varphi^{-1}(x).\partial_\mu\varphi(x).$$

Can we find $\varphi \in \mathrm{Aut}^V(\xi)$ satisfying the condition $\varphi^*(A)_\mu = 0$? The Yang-Mills curvature tensor $\{F_{\mu\nu}\} \neq 0$ is an obstruction to this. If $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] = 0$ we can find (at least locally) an automorphism $\varphi \in \mathrm{Aut}^V(\xi)$ such that

$$\varphi^*(A)_\mu(x) = \varphi^{-1}(x).A_\mu(x).\varphi(x) + \varphi^{-1}(x).\partial_\mu\varphi(x) = 0.$$

The Yang-Mills tensor $F^\nabla$ is the curvature of the linear connection $\nabla$.

### 2.5. An Obstruction for a Given Almost Complex Structure to be Diffeomorphic to the Canonical One

Let $M$ be an even dimensional smooth manifold, $\dim M = m = 2l$ and let

$$J_0 = \sum_{i=1}^{l} \left( \mathrm{d}x^i \otimes \frac{\partial}{\partial x^{2+i}} - \mathrm{d}x^{l+i} \otimes \frac{\partial}{\partial x^i} \right)$$

be an almost complex structure on $M$ in a canonical form in local coordinates. Let $J$ be an arbitrary almost complex structure on $M$ [4], $J \in C^\infty(T^*(M) \otimes T(M))$, and $J^2(x) = -\mathbf{1}$. In the same coordinates

$$J(x) = J^\mu_\nu(x)\mathrm{d}x^\nu \otimes \frac{\partial}{\partial x^\mu} \cdot$$

Every diffeomorphism $\varphi \in \mathrm{Diff}(M)$ has a natural action on $C^\infty(T^*(M) \otimes T(M))$. For every $J \in C^\infty(T^*(M) \otimes T(M))$ we have

$$\varphi^*(J)^\mu_\nu(x) = \frac{\partial \varphi^{-1\mu}}{\partial x^\alpha}(\varphi(x)) \frac{\partial \varphi^\beta}{\partial x^\nu}(x) J^\alpha_\beta(\varphi(x)).$$

Does there exist $\varphi \in \mathrm{Diff}(M)$ such that $\varphi^*(J) = J_0$? The nonvanishing Nijenhuis tensor

$$N(J)^\rho_{\mu\nu} = J^\alpha_\mu \partial_\alpha J^\rho_\nu - J^\alpha_\nu \partial_\alpha J^\rho_\mu - J^\rho_\alpha \partial_\mu J^\alpha_\nu + J^\rho_\alpha \partial_\nu J^\alpha_\mu$$

is an obstruction to this. If $N(J) = 0$ then there exists (at least locally) $\varphi \in \mathrm{Diff}(M)$ such that $\varphi^*(J) = J_0$ (see Newlander-Nirenberg theorem [5]). The Nijenhuis tensor can be written as a vector-valued two-form

$$N(J)(X, Y) = [JX, JY] - J[X, JY] - J[JX, Y] - [X, Y].$$

The Nijenhuis tensor $N(J)$ is in some sense the curvature of the almost complex structure $J$.

All the obstructions to the local equivalence of sections in the corresponding fibre bundles with respect to the action of the “functional” group we mentioned in the upper cases can be viewed as “curvatures” in the broad sense of the word. We can consider them as particular examples of a general notion of a curvature (see Section 3) considered as an universal type of obstructions to the local equivalence in terms of jet bundles, jet lifting of the group action and the algebraization of the differential operators. This scheme applied to the case of a superconnection leads to a notion of a curvature of a superconnection which is different from the one, obtained by a purely algebraic analogy [7]. We consider the definition given in Section 4 to be the adequate one.

## 3. General Notion of a Curvature

For a vector bundle $\xi = (E, p, M)$ the adopted coordinates $(x^\mu, u^a)$ on $\xi$ induce the corresponding coordinates $(x^\mu, u^a, u^a_\nu, u_{\nu_1\nu_2}, \ldots, u^a_{\nu_1,\ldots,\nu_k})$, $1 \leq \nu_1 \leq \nu_2 \leq \cdots \leq \nu_j \leq m$, $j = 2, 3, \ldots, k$ in the k-th jet bundle $j^k(\xi)$ of $\xi$, $j^k(\xi) = (E^k, p^k, M)$ for every $k = 1, 2, 3, \ldots$ (see [6, 8]). Every element of the fibre $j^k(\xi)_x$ of $j^k(\xi)$ equals to $j^k(\psi(x))$ – the value at the point $x$ of the $k$-jet of some $\psi \in C^\infty(\xi)$, in coordinates $u^a(j^k(\psi(x)) = \psi^a(x)$, $u^a_{\nu_1\ldots\nu_j}(j^k(\psi(x))) = \partial_{\nu_1}\ldots\partial_{\nu_j}\psi^a(x)$, where $\psi^a$ are the components of the field $\psi$. The mapping $C^\infty(\xi) \longrightarrow C^\infty(j^k(\xi))$ and given in the local coordinate by

$$\psi(x) \longmapsto (\psi^a(x), \partial_\mu\psi^a(x), \ldots, \partial_{\mu_1}\ldots\partial_{\mu_k}\psi^a(x))$$

is the so called jet lifting of the section $\psi$ and plays the role of an universal differential operator of order $k$ (see [6, 8]).

Let $H$ be a "functional" group and $F_h : E \longrightarrow E$, $h \in H$ be a fibre preserving action of the group $H$ on the bundle $\xi = (E, p, M)$. For sake of simplicity we will have in mind only the group $\mathrm{Aut}^V(\xi)$ of vertical automorphisms of $\xi$, or $\mathrm{Diff}(M)$ – the group of diffeomorphisms of $M$. In the coordinates $(x^\mu, u^a)$ the action $F$ of the group $H$ reads

$$F_h(x^\mu, u^a) = (\varphi^\mu_h(x), F^a_h(x, u))$$

where $h \in H$, $\varphi_h : M \longrightarrow M$ is an action of the group $H$ on the base $M$ of the bundle and $F_{h_1}(\varphi_{h_2}(x), F_{h_2}(x, u)) = F_{h_1h_2}(x, u)$.

The group $H$ has a natural action $\tilde{F}$ on $C^\infty$

$$\tilde{F}_h(\psi)(x) = F_h(\psi(\varphi^{-1}_h(x))) \tag{4}$$

where $\psi \in C^\infty(\xi)$, $h \in H$, $x \in M$. The action (4) of the group $H$ on $C^\infty(\xi)$ induces an action of $H$ on the jet-bundle $j^k(\xi)$ for every $k = 1, 2, 3, \ldots$ .

Let $\psi_1$ and $\psi_2$ be two sections of the bundle $\xi$. The problem we deal with is the (local) equivalence of $\psi_1$ and $\psi_2$ with respect to the group $H$, i.e., the existence of an element $h \in H$ such that

$$\tilde{F}_h(\psi_2) = \psi_1. \tag{5}$$

In other words for given $\psi_1$ and $\psi_2$, the expression (5) is an equation for $h \in H$. If the equation (5) is satisfied then its $k$-jet lifting

$$\tilde{F}_h(j^k(\psi_2)) = j^k(\psi_1) \tag{6}$$

is also satisfied for every $k$. Inversely, if we can proof that for some $k$ the equation (6) has no solution for $h \in H$ then obviously this is an obstruction to the solvability of (5), i.e., to the local equivalence of $\psi_1$ and $\psi_2$. The equation (6) involves the derivatives of $\psi_1$ and $\psi_2$ up to the order $k$. Therefore, it is easier to deal with (6).

We begin with the study of the condition imposed by the equation (6) on the $k$-jets of $\psi_1$ and $\psi_2$ at a point $x_0 \in M$.

Let

$$G_{x_0} := \{h \in H\,;\, \varphi_h(x_0) = x_0\}$$

be the stationary group of the point $x_0$. The group $G_{x_0}$ has a natural action on $j^k(\xi)_{x_0}$ for every $k$. Let us consider first the case $k = 0$. The space $j^0(\xi)_{x_0}$ is simply the fibre $\xi_{x_0}$. For $h \in G_{x_0}$ the equation (6) leads to

$$\tilde{F}_h(\psi_2(x_0)) = \psi_1(x_0). \tag{7}$$

Insolvability of (7) with respect to $h$ means that $\psi_1(x_0)$ and $\psi_2(x_0)$ belong to different orbits of $G_{x_0}$ in $\xi_{x_0}$. If the fibre $\xi_{x_0}$ is a homogeneous space for the group $G_{x_0}$ there is no obstruction to the solvability of (6) arising from the equation (7). Then looking for an obstruction we go to $k = 1$

$$\tilde{F}_h(j^1(\psi_2)_{x_0}) = j^1(\psi_1)_{x_0}. \tag{8}$$

If the fibre $j^1(\xi)_{x_0}$ is again a homogeneous space for the group $G_{x_0}$ there is no an obstruction to the solvability of equation (6). We proceed to the first $k$ for which the fibre $j^k(\xi)_{x_0}$ is not a homogeneous space for the group $G_{x_0}$. Let $\tilde{\pi}_{x_0}$ be the canonical projection to the factor space $j^k(\xi)_{x_0}/G_{x_0}$

$$\tilde{\pi}_{x_0} : j^k(\xi)_{x_0} \longrightarrow j^k(\xi)_{x_0}/G_{x_0}. \tag{9}$$

If

$$\tilde{\pi}_{x_0}(j^k(\psi_2)_{x_0}) \neq \tilde{\pi}_{x_0} j^k(\psi_1)_{x_0}) \tag{10}$$

then the $k$-jets $j^k(\psi_2)_{x_0}$ and $j^k(\psi_1)_{x_0}$ belong to different orbits of the group $G_{x_0}$ and the equation

$$\tilde{F}_h(j^k(\psi_2))_{x_0} = j^k(\psi_1)_{x_0} \tag{11}$$

has no solution with respect to $h \in G_{x_0}$. The condition (11) is an obstruction to the local equivalence of the sections $\psi_2$ and $\psi_1$ in a neighborhood of the point $x_0$. These considerations are valid for every point $x \in M$ and we obtain a fibre preserving map (over the identity)

$$\tilde{\pi} : j^k(\xi) \longrightarrow j^k(\xi)/H$$

where the fibre of $j^k(\xi)/H$ at a point $x \in M$ is the factor space $j^k(\xi)_x/G_x$. The fibre preserving map $\tilde{\pi}$ is the symbol of the differential operator

$$\pi : C^\infty(\xi) \longrightarrow C^\infty(j^k(\xi)/H)$$

i.e., $\pi(\psi) = \tilde{\pi}(j^k(\psi))$. The condition $\pi(\psi_2) \neq \pi(\psi_1)$ is an obstruction to the (local) equivalence. Therefore, we consider the differential operator $\pi$ as a curvature – the general notion of curvature that we propose in this paper ($\pi(\psi)$ is the curvature of the section $\psi$).

One can show that the classical results listed in Section 2 are explicit examples of the differential operator $\pi$. Namely:

2.1 The sections we consider are the two forms $\omega$ on $M$, the functional group is the group $\mathrm{Diff}(M)$, $k = 1$ and $\pi(\omega) = \mathrm{d}\omega$.

2.2 The sections we consider are the metrics $g$ on $M$, the functional group is the group $\mathrm{Diff}(M)$, $k = 2$ and $\pi(g) = R(g)$ – the Riemannian curvature tensor.

2.3 The sections we consider are the conformaly equivalent classes $[g]$ of metrics $g$, the functional group is $\mathrm{Diff}(M)$, $k = 2$ and $\pi([g]) = W(g)$ – the Weil conformal tensor.

2.4 The sections we consider are the linear connections $\nabla$ on a vector bundle $\xi$, $k = 1$ and $\pi(\nabla) = F^\nabla$ – the Yang-Mills curvature tensor.

2.5 The sections we consider are the almost complex structures $J$ on $M$, the functional group is $\mathrm{Diff}(M)$, $k = 1$ and $\pi(J) \sim N(J)$ – the Nijenhuis tensor.

In the next section we will consider the cases 2.4 and 2.5.

**Remark.** Here we consider two kinds of groups:

1. The group $H$ is the group of vertical automorphisms of a vector bundle $\xi$, $H = \mathrm{Aut}^V(\xi)$. In this case the stationary group of a point $x_0$, $G_{x_0} = H$.
2. The group $H$ is the group of the diffeomorphisms of a manifold $M$, $H = \mathrm{Diff}(M)$. The bundle $\xi$ is some tensor power of $T(M)$ and of $T^*(M)$ and the action of $H$ is its tangent lifting. In this case it is enough to consider only its stationary group $G_x$ at each point $x$. This matter will be discussed elsewhere.

## 4. Examples

### Example 2.4. Linear Connections on a Vector Bundle

We will describe the curvature of a linear connection on a vector bundle as an obstruction to the local equivalence of linear connections in terms of the general scheme given in Section 3.

For every linear connection $\nabla$ on a vector bundle $\xi$ we have $\nabla_\mu = \partial_\mu + A_\mu$, where $\nabla_{0\mu} = \partial_\mu$ is the "origin" and $\{A_\mu\} \equiv A \in \Omega^1(\xi^* \otimes \xi)$. The elements of $\Omega^1(\xi^* \otimes \xi)$ are in one-to-one correspondence with the linear connections on the bundle $\xi$ and we recognize $\Omega^1(\xi^* \otimes \xi)$ as the space of sections of $T^*(M) \otimes \xi^* \otimes \xi$, where the "functional" group $\mathrm{Aut}^V(\xi)$ acts. To accomplish the procedure described in Section 3, we choose coordinates $\{x^\mu, u^a\}$ to be centered at a point $x_0 \in M$, $x^\mu(x_0) = 0$. The first jet of $A_\mu$ at the point 0 reads

$$j^1(A_\mu)_0(x) = \mathbf{A}_\mu + \mathbf{A}_{\mu\alpha} x^\alpha, \qquad \mathbf{A}_\mu = A_\mu(0), \qquad \mathbf{A}_{\mu\alpha} = \partial_\alpha A_\mu(0) \tag{12}$$

where $\mu, \alpha = 1, 2, \ldots, m$. The set of all pairs of arbitrary matrices $(\{\mathbf{A}_\mu\}, \{\mathbf{A}_{\mu\alpha}\})$ describes $j^1(T^*(M) \otimes \xi^* \otimes \xi)_0$. Let $\varphi \in \mathrm{Aut}^V(\xi)$ be a vertical automorphism, $\varphi(x^\mu) \in \mathrm{GL}(n, \mathbb{R})$. The second jet of $\varphi$ at the point 0 reads

$$j^2(\varphi)_0(x) = \mathbf{B}_0 + \mathbf{B}_\alpha x^\alpha + \frac{1}{2}\mathbf{B}_{\alpha\beta} x^\alpha x^\beta, \quad \mathbf{B}_0 = \varphi(0), \quad \mathbf{B}_{\alpha\beta} = \partial_\alpha \partial_\beta \varphi(0) \tag{13}$$

where $\alpha, \beta = 1, 2, \ldots, m$. The set of all triples of arbitrary matrices $\{\mathbf{B}_0$ with $\det(\mathbf{B}_0) \neq 0$, $\mathbf{B}_\alpha$, $\mathbf{B}_{\alpha\beta}$ symmetric with respect to $\alpha, \beta\}$ describes the space of 2-jets of the vertical automorphisms at the point 0. The matrix $\mathbf{B}_0 = \varphi(0)$ is non-degenerate and can be considered as a common multiplier of the entire 2-jet and it does not play an essential role in our considerations. It is enough to consider only automorphisms $\varphi$ with 2-jets of the form

$$j^2(\varphi)_0(x) = \mathbf{1} + \mathbf{B}_\alpha x^\alpha + \frac{1}{2}\mathbf{B}_{\alpha\beta} x^\alpha x^\beta. \tag{14}$$

The action of $\varphi$ by its two-jet (14) on $j^1(T^*(M) \otimes \xi^* \otimes \xi)_0$ is given by

$$j^1(A_\mu)_0 \mapsto j^1(\varphi^*(A)_\mu)_0 = j^1(\varphi^{-1})_0 . j^1(A_\mu)_0 . J^1(\varphi)_0 + j^1(\varphi^{-1})_0 . j^1(\partial_\mu \varphi)_0$$

$$\left| \begin{array}{l} \mathbf{A}_\mu \mapsto \mathbf{A}_\mu + \mathbf{B}_\mu \\ \mathbf{A}_{\mu\alpha} \mapsto \mathbf{A}_{\mu\alpha} + \mathbf{A}_\mu . \mathbf{B}_\alpha - \mathbf{B}_\alpha . \mathbf{A}_\mu - \mathbf{B}_\alpha . \mathbf{B}_\mu + \mathbf{B}_{\mu\alpha}. \end{array} \right. \tag{15}$$

The formulae (15) describe the action of $(\mathbf{1}, \mathbf{B}_\alpha, \mathbf{B}_{\mu\alpha})$ on the space $(\{\mathbf{A}_\mu\}, \{\mathbf{A}_{\mu\alpha}\})$. Our purpose is to describe explicitly the projection

$$\tilde{\pi} : j^1(T^*(M) \otimes \xi^* \otimes \xi)_0 \longrightarrow j^1(T^*(M) \otimes \xi^* \otimes \xi))_0 / \mathrm{Aut}^V(\xi). \tag{16}$$

First of all $j^0(T^*(M) \otimes \xi^* \otimes \xi)_0$ is a homogeneous space. So we can take $\mathbf{B}_\mu = -\mathbf{A}_\mu$ and acting by $(\mathbf{1}, -\mathbf{A}_\mu, \mathbf{B}_{\mu\alpha})$ on $j^1(A_\mu)_0$ we obtain $(\mathbf{A}_\mu, \mathbf{A}_{\mu\alpha}) \mapsto (0, \tilde{\mathbf{A}}_{\mu\alpha})$, where $\tilde{\mathbf{A}}_{\mu\alpha} = \mathbf{A}_{\mu\alpha} - \mathbf{A}_\mu \mathbf{A}_\alpha + \mathbf{B}_{\mu\alpha}$. Due to the homogeneity of $j^0(T^*(M) \otimes \xi^* \otimes \xi)_0$ there is no obstruction at the level of the zero-jets and we go to the level of first jets. So we consider only elements of the type $(\mathbf{1}, 0, \mathbf{B}_{\mu\alpha})$ acting on $(0, \tilde{\mathbf{A}}_{\mu\alpha})$, i.e., $\tilde{\mathbf{A}}_{\mu\alpha} \mapsto \tilde{\mathbf{A}}_{\mu\alpha} + \mathbf{B}_{\mu\alpha}$.

Due to the symmetry of the matrices $\mathbf{B}_{\mu\alpha}$ with respect to $\mu$, $\alpha$ the factor space $(\{\mathbf{A}_{\mu\alpha}\})/(\{\mathbf{B}_{\mu\alpha}\})$ is represented by the space of matrices $\{\mathbf{A}_{\mu\alpha}\}$ antisymmetric with respect to $\mu$, $\alpha$ and the projection is the antisymmetrization with respect to $\mu$, $\alpha$: $\tilde{\mathbf{A}}_{\mu\alpha} \mapsto \tilde{\mathbf{A}}_{\alpha\mu} - \tilde{\mathbf{A}}_{\mu\alpha}$. Finally we obtain

$$(\mathbf{A}_\mu, \mathbf{A}_{\mu\alpha}) \mapsto (0, \tilde{\mathbf{A}}_{\mu\alpha}) \mapsto \tilde{\mathbf{A}}_{\alpha\mu} - \tilde{\mathbf{A}}_{\mu\alpha}$$

$$\tilde{\mathbf{A}}_{\alpha\mu} - \tilde{\mathbf{A}}_{\mu\alpha} = \partial_\mu \tilde{\mathbf{A}}_\alpha - \partial_\alpha \tilde{\mathbf{A}}_\mu = \partial_\mu A_\alpha(0) - \partial_\alpha A_\mu(0) + [A_\mu(0), A_\alpha(0)] = F^\nabla_{\mu\nu}(0). \tag{17}$$

The differential operator $\pi$ corresponding to $\tilde{\pi}$ (16) is the Yang-Mills curvature tensor

$$\pi(\nabla) = F^\nabla.$$

For a flat connection the curvature tensor is equal to zero. For a connection $\nabla$ the condition

$$F^\nabla \equiv \tilde{\pi}(j^1(A)) \neq 0$$

is an obstruction for $\nabla$ to be a flat connection.

### Example 2.5. Almost Complex Structure on an Even Dimensional Manifold

Following the general scheme considered in Section 3 we will describe the Nijenhuis differential operator as an obstruction to the (local) equivalence of almost complex structures on the even dimensional manifold $M$. The nonvanishing Nijenhuis tensor for some almost complex structure is an obstruction to its integrability, i.e., to its equivalence to the canonical almost complex structure for which the Nijenhuis tensor vanishes.

In this example the bundle under consideration is the vector bundle $T^*(M) \otimes T(M)$. An almost complex structure $J$ on $M$ is an element of $C^\infty(T^*(M) \otimes T(M))$ with the property $J^2 = -\mathbf{1}$. In coordinates $\{x^\mu\}$ centered at a point $x_0 \in M$, $x^\mu(x_0) = 0$, the first jet of an almost complex structure $J$ reads

$$j^1(J)_0{}^\mu_\nu(x) = \mathbf{J}^\mu_\nu + \mathbf{C}^\mu_{\nu\rho} x^\rho$$

where $\mathbf{J}^\nu_\mu = J^\nu_\mu(0)$, $\mathbf{C}^\mu_{\nu\rho} = \partial_\rho J^\mu_\nu(0)$, $\mathbf{J}^\mu_\alpha \mathbf{J}^\alpha_\nu = -\delta^\mu_\nu$, and $\mathbf{J}^\mu_\alpha \mathbf{C}^\alpha_{\nu\rho} + \mathbf{C}^\mu_{\alpha\rho} \mathbf{J}^\alpha_\mu = 0$.

The functional group is the group $\mathrm{Diff}(M)$ acting by the tangent lifting. The stationary group is the group $\mathrm{Diff}(M)_{x_0}$ – the group of the diffeomorphisms with a stable point $x_0$. For the second jet of a diffeomorphism $\varphi \in \mathrm{Diff}(M)_{x_0}$ we have

$$j^2(\varphi)^\mu_0(x) = \mathbf{B}^\mu_\alpha x^\alpha + \frac{1}{2} \mathbf{B}^\mu_{\alpha\beta} x^\alpha x^\beta$$

where $\mathbf{B}^\mu_\alpha = \partial_\alpha \varphi^\mu(0)$, $\mathbf{B}^\mu_{\alpha\beta} = \partial_\alpha \partial_\beta \varphi^\mu(0)$ and $\det\{\mathbf{B}^\mu_\alpha\} \neq 0$.

Here again the crucial role play the diffeomorphisms $\varphi$ with $\mathbf{B}^\mu_\alpha = \delta^\mu_\alpha$, i.e., with a tangent lifting $\varphi^T : T_{x_0}(M) \longrightarrow T_{x_0}(M)$ equal to the identity map. We will consider only diffeomorphisms of this kind. For the 2-jets we have

$$j^2(\varphi_0)^\mu(x) = x^\mu + \frac{1}{2} \mathbf{B}^\mu_{\alpha\beta} x^\alpha x^\beta.$$

The action of these diffeomorphisms on the first jet of the almost complex structure $J$ at the point 0 is given by

$$\left| \begin{array}{l} \mathbf{J}^{\mu}_{\nu} \longrightarrow \mathbf{J}^{\mu}_{\nu} \\ \mathbf{C}^{\mu}_{\nu\rho} \longrightarrow \mathbf{C}^{\mu}_{\nu\rho} + \mathbf{J}^{\mu}_{\alpha}\mathbf{B}^{\alpha}_{\nu\rho} - \mathbf{B}^{\mu}_{\alpha\rho}\mathbf{J}^{\alpha}_{\nu}. \end{array} \right. \tag{18}$$

Due to (18) we consider only the space $W$ of 1-jets of the almost complex structure $J$ at the point 0 over a fixed $\{\mathbf{J}^{\mu}_{\nu}\} \equiv \mathbf{J}$. This space is parameterized by $\{\mathbf{C}^{\mu}_{\nu\rho}\}$. The description of the space $W$ and of the factor-space of $W$ with respect to the action (18) is more visual if we consider $\{\mathbf{C}^{\mu}_{\nu\rho}\}$ and $\{\mathbf{B}^{\mu}_{\alpha\beta}\}$ as elements of the space $L^* \otimes L^* \otimes L$, where $L$ is a vector space with $\dim(L) = m$, or equivalently as bilinear forms: $L \times L \longrightarrow L$.

In this interpretation

$$W = \{\mathbf{C} \in L^* \times L^* \times L;\, \mathbf{J}\mathbf{C}(u,v) + \mathbf{C}(\mathbf{J}u,v) = 0,\ u,v \in L\}.$$

Let us define a map $K : S^2L^* \otimes L \longrightarrow W$ by

$$K(\mathbf{B})(u,v) = \mathbf{J}\mathbf{B}(u,v) - \mathbf{B}(\mathbf{J}u,v). \tag{19}$$

In theses notations the action (18) of the diffeomorphisms on the space $W$ reads

$$W \ni \mathbf{C} \longrightarrow \mathbf{C} + K(\mathbf{B}), \qquad \mathbf{B} \in S^2L^* \otimes L.$$

Our purpose is to describe the projection

$$\pi : W \longrightarrow W/K(S^2L^* \otimes L).$$

By $s : W \longrightarrow S^2L^* \otimes L$ we denote the symmetrization

$$s(\mathbf{C})(u,v) = \frac{1}{2}(\mathbf{C}(u,v) + \mathbf{C}(v,u)).$$

The map $A \equiv s \circ K : S^2L^* \otimes L \longrightarrow S^2L^* \otimes L$ is an invertible map but not equal to the identity. The space $W$ splits into the following direct sum

$$W = K(S^2L^* \otimes L) \oplus \ker(s). \tag{20}$$

The projections on the first and on the second term in (20) are the following

$$K \circ A^{-1} \circ s : W \longrightarrow K(S^2L^* \otimes L)$$

and

$$\mathbf{1} - K \circ A^{-1} \circ s : W \longrightarrow \ker(s).$$

It is easy to calculate that

$$(K \circ A^{-1} \circ s)(\mathbf{C})(u,v) = \frac{1}{2}(\mathbf{C}(u,v) + \mathbf{C}(v,u) - \mathbf{J}\mathbf{C}(u,\mathbf{J}v) + \mathbf{J}\mathbf{C}(v,\mathbf{J}u))$$

and

$$(\mathbf{1} - K \circ A^{-1} \circ s)(\mathbf{C})(u,v)$$
$$= \tfrac{1}{2}(\mathbf{C}(u,v) - \mathbf{C}(v,u) - \mathbf{JC}(u,\mathbf{J}v) + \mathbf{JC}(v,\mathbf{J}u)) = -\tfrac{1}{2}\mathbf{J}.\mathbf{N}(J).$$

The projection

$$-\frac{1}{2}\mathbf{J}.N(\mathbf{J}) : W \longrightarrow \ker(s) \approx W/K(S^2L^* \otimes L)$$

is the projection we are looking for. The operator $J(x)$ is an invertible operator so the important information is carried by $N(J)$. For the canonical almost complex structure $N(J) = 0$. That is why $N(J) \neq 0$ is an obstruction to the (local) equivalence of the almost complex structure $J$ to the canonical one, i.e., to its integrability.

## 5. Definition of the Curvature of a Superconnection as an Obstruction

Let $\nabla_s = \nabla + \chi$ be a superconnection [7] on a $\mathbb{Z}_2$-graded bundle $\xi$. Let $\{x^\mu\}$ be coordinates on the base $M$ and $\{e_{+a}\}$, $\{e_{-i}\}$ be a basis in $\xi$, compatible with the $\mathbb{Z}_2$-grading. In coordinates

$$\begin{gathered}\nabla_\mu = \partial_\mu + A_\mu, \qquad A_\mu = A_{+\mu} + A_{-\mu}\\ A_+ = A_{+\mu a}^{\ \ b}\,\mathrm{d}x^\mu \otimes e_+^a \otimes e_{+b}, \qquad A_- = A_{-\mu i}^{\ \ j}\,\mathrm{d}x^\mu \otimes e_-^{\ i} \otimes e_{-j}\\ \chi = \chi_{+-}{}_i^a\, e_-^{\ i} \otimes e_{+a} + \chi_{-+}{}_a^i\, e_+^{\ a} \otimes e_{-i}\\ (\chi_{+-} + \chi_{-+}, A_+ + A_-) \in \Omega^{0,1}(\xi^* \otimes \xi)_-.\end{gathered} \tag{21}$$

If the origin $\nabla_{s0\mu} = \partial_\mu + 0$ in the space of the superconnections is fixed the elements of $\Omega^{0,1}(\xi^* \otimes \xi)_-$ are in one-to-one correspondence with the superconnections. So in the case of superconnections the bundle under consideration is $(\xi^* \otimes \xi)_- \oplus (T^*(M) \otimes \xi^* \otimes \xi)_-$. The first jet at the point 0 of a superconnection, or more precisely, of its components reads

$$\begin{aligned}
j^1(\chi_{+-})_0(x) &= \boldsymbol{\chi}_{+-} + \boldsymbol{\chi}_{+-\mu}x^\mu, & \boldsymbol{\chi}_{+-} &= \chi_{+-}(0), & \boldsymbol{\chi}_{+-\mu} &= \partial_\mu \chi_{+-}(0)\\
j^1(\chi_{-+})_0(x) &= \boldsymbol{\chi}_{-+} + \boldsymbol{\chi}_{-+\mu}x^\mu, & \boldsymbol{\chi}_{-+} &= \chi_{-+}(0), & \boldsymbol{\chi}_{-+\mu} &= \partial_\mu \chi_{-+}(0)\\
j^1(A_{+\mu})_0(x) &= \mathbf{A}_{+\mu} + \mathbf{A}_{+\mu\rho}x^\rho, & \mathbf{A}_{+\mu} &= A_{+\mu}(0), & \mathbf{A}_{+\mu\rho} &= \partial_\rho A_{+\mu}(0)\\
j^1(A_{-\mu})_0(x) &= \mathbf{A}_{-\mu} + \mathbf{A}_{-\mu\rho}x^\rho, & \mathbf{A}_{-\mu} &= A_{-\mu}(0), & \mathbf{A}_{-\mu\rho} &= \partial_\rho A_{-\mu}(0).
\end{aligned}$$

The set $(\boldsymbol{\chi}_{+-}, \boldsymbol{\chi}_{+-\rho}, \boldsymbol{\chi}_{-+}, \boldsymbol{\chi}_{-+\rho}, \mathbf{A}_{+\mu}, \mathbf{A}_{+\mu\rho}, \mathbf{A}_{-\mu}, \mathbf{A}_{-\mu\rho})$ parameterizes the jet-space $j^1((\xi^* \otimes \xi)_- \oplus (T^*(M) \otimes \xi^* \otimes \xi)_-)_0$.

The "functional" group is $\mathrm{Aut}^V(\xi_+ \oplus \xi_-)$. For every element $\varphi \in \mathrm{Aut}^V(\xi_+ \oplus \xi_-)$ we have $\varphi = \varphi_+ + \varphi_-$. The action of $\varphi$ on $\Omega^{0,1}(\xi^* \otimes \xi)$ is given by

$$(\chi_{+-} + \chi_{-+}, A_+ + A_-) \longrightarrow \varphi^*(\chi_{+-} + \chi_{-+}, A_+ + A_-) = (\varphi_+^{-1}.\chi_{+-}.\varphi_- \\ + \varphi_-^{-1}.\chi_{-+}.\varphi_+, \varphi_+^{-1}.A_{+\mu}.\varphi_+ + \varphi_+^{-1}.\partial_\mu\varphi_+, \varphi_-^{-1}.A_{-\mu}.\varphi_- + \varphi_-^{-1}.\partial_\mu\varphi_-) \quad (22)$$

The second jet of $\varphi \in \mathrm{Aut}^V(\xi_+ \oplus \xi_-)$ at the point 0 reads

$$j^2(\varphi_+)(x) = \mathbf{1} + \boldsymbol{\varphi}_{+\rho}x^\rho + \frac{1}{2}\boldsymbol{\varphi}_{+\rho\sigma}x^\rho x^\sigma, \boldsymbol{\varphi}_{+\rho} = \partial_\rho\varphi_+(0), \boldsymbol{\varphi}_{+\rho\sigma} = \partial_\rho\partial_\sigma\varphi_+(0)$$
$$j^2(\varphi_-)(x) = \mathbf{1} + \boldsymbol{\varphi}_{-\rho}x^\rho + \frac{1}{2}\boldsymbol{\varphi}_{-\rho\sigma}x^\rho x^\sigma, \boldsymbol{\varphi}_{-\rho} = \partial_\rho\varphi_-(0), \boldsymbol{\varphi}_{-\rho\sigma} = \partial_\rho\partial_\sigma\varphi_-(0).$$

As in the previous examples (see Section 4) we consider only automorphisms with 2-jets beginning with the identity operator. From (21) and (22) for the action of an automorphism $\varphi$ on the first jet of the superconnection we obtain

$$\begin{array}{|lll} \boldsymbol{\chi}_{+-} & \longrightarrow & \boldsymbol{\chi}_{+-} \\ \boldsymbol{\chi}_{+-\mu} & \longrightarrow & \boldsymbol{\chi}_{+-\mu} - \boldsymbol{\varphi}_{+\mu}.\boldsymbol{\chi}_{+-} + \boldsymbol{\chi}_{+-}.\boldsymbol{\varphi}_{-\mu} \end{array}$$
$$\begin{array}{|lll} \boldsymbol{\chi}_{-+} & \longrightarrow & \boldsymbol{\chi}_{-+} \\ \boldsymbol{\chi}_{-+\mu} & \longrightarrow & \boldsymbol{\chi}_{-+\mu} - \boldsymbol{\varphi}_{-\mu}.\boldsymbol{\chi}_{-+} + \boldsymbol{\chi}_{-+}.\boldsymbol{\varphi}_{+\mu} \end{array}$$
$$\begin{array}{|lll} \mathbf{A}_{+\mu} & \longrightarrow & \mathbf{A}_{+\mu} + \boldsymbol{\varphi}_{+\mu} \\ \mathbf{A}_{+\mu\rho} & \longrightarrow & \mathbf{A}_{+\mu\rho} - \boldsymbol{\varphi}_{+\rho}.\mathbf{A}_{+\mu} + \mathbf{A}_{+\mu}.\boldsymbol{\varphi}_{+\rho} - \boldsymbol{\varphi}_{+\rho}.\boldsymbol{\varphi}_{+\mu} + \boldsymbol{\varphi}_{+\mu\rho} \end{array} \quad (23)$$
$$\begin{array}{|lll} \mathbf{A}_{-\mu} & \longrightarrow & \mathbf{A}_{-\mu} + \boldsymbol{\varphi}_{-\mu} \\ \mathbf{A}_{-\mu\rho} & \longrightarrow & \mathbf{A}_{-\mu\rho} - \boldsymbol{\varphi}_{-\rho}.\mathbf{A}_{-\mu} + \mathbf{A}_{-\mu}.\boldsymbol{\varphi}_{-\rho} - \boldsymbol{\varphi}_{-\rho}.\boldsymbol{\varphi}_{-\mu} + \boldsymbol{\varphi}_{-\mu\rho}. \end{array}$$

We choose $\boldsymbol{\varphi}_{+\mu} = -\mathbf{A}_{+\mu}$, $\boldsymbol{\varphi}_{-\mu} = -\mathbf{A}_{-\mu}$ and the upper transformations lead to

$$(\boldsymbol{\chi}_{+-}, \boldsymbol{\chi}_{+-\mu}, \ \boldsymbol{\chi}_{-+}, \boldsymbol{\chi}_{-+\mu}, \mathbf{A}_{+\mu}, \mathbf{A}_{+\mu\rho}, \mathbf{A}_{-\mu}, \mathbf{A}_{-\mu\rho}) \longrightarrow \\ (\boldsymbol{\chi}_{+-}, \tilde{\boldsymbol{\chi}}_{+-\mu}, \boldsymbol{\chi}_{-+}, \tilde{\boldsymbol{\chi}}_{-+\mu}, 0, \tilde{\mathbf{A}}_{+\mu\rho}, 0, \tilde{\mathbf{A}}_{\mu\rho}) \quad (24)$$

$$\begin{array}{|ll} \tilde{\boldsymbol{\chi}}_{+-\mu} = & \boldsymbol{\chi}_{+-\mu} + \mathbf{A}_{+\mu}.\boldsymbol{\chi}_{+-} - \boldsymbol{\chi}_{+-}.\mathbf{A}_{-\mu} \\ \tilde{\boldsymbol{\chi}}_{-+\mu} = & \boldsymbol{\chi}_{-+\mu} + \mathbf{A}_{-\mu}.\boldsymbol{\chi}_{-+} - \boldsymbol{\chi}_{-+}.\mathbf{A}_{+\mu} \\ \tilde{\mathbf{A}}_{+\mu\rho} = & \mathbf{A}_{+\mu\rho} - \mathbf{A}_{+\mu}.\mathbf{A}_{+\rho} \\ \tilde{\mathbf{A}}_{-\mu\rho} = & \mathbf{A}_{-\mu\rho} + \mathbf{A}_{-\mu}.\mathbf{A}_{-\rho}. \end{array}$$

Next we consider automorphisms which acting according (23) preserve the special form (24) of the first jet of the superconnection. The first jets of these automorphisms have the form $(\mathbf{1}, 0, \boldsymbol{\varphi}_{+\mu\rho}; \mathbf{1}, 0, \boldsymbol{\varphi}_{-\mu\rho})$. Their action on

$$(\boldsymbol{\chi}_{+-}, \tilde{\boldsymbol{\chi}}_{+-\rho}, \boldsymbol{\chi}_{-+}, \tilde{\boldsymbol{\chi}}_{-+\rho}, 0, \tilde{\mathbf{A}}_{+\mu\rho}, 0, \tilde{\mathbf{A}}_{-\mu\rho})$$

is given by

$$(\boldsymbol{\chi}_{+-}, \tilde{\boldsymbol{\chi}}_{+-\rho}, \boldsymbol{\chi}_{-+}, \tilde{\boldsymbol{\chi}}_{-+\rho}, 0, \tilde{A}_{+\mu\rho}, 0, \tilde{A}_{-\mu\rho}) \\ \longrightarrow (\boldsymbol{\chi}_{+-}, \tilde{\boldsymbol{\chi}}_{+-\rho}, \boldsymbol{\chi}_{-+}, \tilde{\boldsymbol{\chi}}_{-+\rho}, 0, \tilde{\mathbf{A}}_{+\mu\rho} + \boldsymbol{\varphi}_{+\mu\rho}, 0, \tilde{\mathbf{A}}_{-\mu\rho} + \boldsymbol{\varphi}_{-\mu\rho}).$$

Due to the symmetry of the matrices $\boldsymbol{\varphi}_{\pm\mu\rho}$ with respect to $\mu$, $\rho$ the factor space, we are looking for, is obtained by antisymmetrization of $\{\tilde{\mathbf{A}}_{\pm\mu\rho}\}$. The canonical projection

$$\tilde{\pi} : j^1((\xi^* \otimes \xi)_- \oplus (T^*(M) \otimes \xi^* \otimes \xi)_-) \\ \longrightarrow j^1((\xi^* \otimes \xi)_- \oplus (T^*(M) \otimes \xi^* \otimes \xi)_-)/\operatorname{Aut}^V(\xi_+ \oplus \xi_-)$$

is the composition of (24) and the antisymmetrization of $\{\tilde{\mathbf{A}}_{\pm\mu\rho}\}$ with respect to $\mu$, $\rho$

$$\begin{aligned} &(\boldsymbol{\chi}_{+-}, \boldsymbol{\chi}_{+-\mu}, \boldsymbol{\chi}_{-+}, \boldsymbol{\chi}_{-+\mu}, \mathbf{A}_{+\mu}, \mathbf{A}_{+\mu\rho}, \mathbf{A}_{-\mu}, \mathbf{A}_{-\mu\rho}) \\ &\longrightarrow (\boldsymbol{\chi}_{+-}, \tilde{\boldsymbol{\chi}}_{+-\mu}, \boldsymbol{\chi}_{-+}, \tilde{\boldsymbol{\chi}}_{-+\mu}, 0, \tilde{\mathbf{A}}_{+\mu\rho}, 0, \tilde{\mathbf{A}}_{-\mu\rho}) \\ &\longrightarrow (\boldsymbol{\chi}_{+-}, \tilde{\boldsymbol{\chi}}_{+-\mu}, \boldsymbol{\chi}_{-+}, \tilde{\boldsymbol{\chi}}_{-+\mu}, 0, \tilde{\mathbf{A}}_{+\rho\mu} - \tilde{\mathbf{A}}_{+\mu\rho}, 0, \tilde{\mathbf{A}}_{-\rho\mu} - \tilde{\mathbf{A}}_{-\mu\rho}) \\ &= (\chi_{+-}(0), \nabla_\mu \chi_{+-}(0), \chi_{-+}(0), \nabla_\mu \chi_{-+}(0), \partial_\mu A_{+\rho}(0) - \partial_\rho A_{+\mu}(0) \\ &\quad + [A_{+\mu}(0), A_{+\rho}(0)], \partial_\mu A_{-\rho}(0) - \partial_\rho A_{-\mu}(0) + [A_{-\mu}(0), A_{-\rho}(0)]). \end{aligned}$$

In short notations

$$\tilde{\pi} : j^1(\chi, \nabla) \longrightarrow (\chi, \nabla\chi, F^\nabla) \in \Omega^{0,1,2}(\xi^* \otimes \xi).$$

Finally, we claim that the natural notation of curvature of a superconnection $\nabla_s = \nabla + \chi$ on a $\mathbb{Z}_2$-graded vector bundle $\xi$ is the operator

$$\pi(\nabla_s) = (\chi, \nabla(\chi), F^\nabla) \in \Omega^{0,1,2}(\xi^* \otimes \xi).$$

This notion differs from the expression $(\chi^2, \nabla(\chi), F^\nabla)$ obtained by purely algebraic analogy with the curvature of a linear connection.

Our notion of a curvature of a superconnection may be of interest to the models of interacting particles in the supersymmetrical field theories where one of the expressions in the Lagrangian is the square of the supercurvature.

The obstructions we have considered are related to the action of the "functional" groups on $k$-jets of smooth sections of some vector bundles. The study of the orbits of this action is usually called a study of the singularities of smooth maps or "Catastrophe Theory" in the terminology of René Thom (see [1, 3]). The title of our paper is in the René Thom's terminology.

## References

[1] Arnol'd V., Varchenko A. and Gusejn-Zade S., *Osobenosti Differenciruemyh Otobrazhenij*, vols 1 & 2, Nauka, Moskwa, 1982 and 1984.

[2] Besse A., *Einstein Manifolds*, Springer, Berlin, 1987.

[3] Brocker T. and Lander L., *Differentiable Germs and Catastrophes*, Cambridge University Press, Cambridge, 1975.

[4] Kolar I., Michor P. and Slovak J., *Natural Operators in Differential Geometry*, Springer, Berlin, 1993.

[5] Newlander A., Nirenberg L., *Complex Analytic Coordinates in Almost Complex Manifold*, Ann. Math **65** (1954) 391–404.

[6] Palais R., *Seminar on the Atiyah-Singer Index Theorem*, Princeton, New Jersey, 1965.

[7] Quillen D., *Superconnections and the Chern Characters*, Topology **24** (1985) 89–95.

[8] Saunders D., *The Geometry of Jet Bundles*, Cambridge University Press, Cambridge, 1989.