Ratner's measure classification theorem: the semisimple case

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These are brief notes to accompany the talk on Ratner's Theorems, as part of the Dynamics working seminar in Herbstemester 2024.

Basically everything here is taken (almost verbatim) from the book of Manfred Einsiedler and Thomas Ward [2], all mistakes here are of course introduced by the author and the associated blame lies solely with him.

Most noticeable is the lack of diagrams to help explain the arguments (especially Lemma 2.3 and Proposition 2.6), for these look at [2, Figures 6.2 and 6.3].

Contents

1 Introduction

Let G be a connected Lie Group, and $\Gamma \leq G$ a lattice[1], and consider the homogeneous space $X = \Gamma\backslash G$.

We aim to study measures on X that are invariant under certain types of flows. In particular, we will need to study orbits of different (but nearby) points in X under this flow.

Suppose that our flow is given by a one-parameter subgroup $B = \{b_t \mid t \in \mathbb{R}\}$, and consider the orbits of x and $y = \varepsilon \cdot x$ (where ε is small, and is the local displacement between x and y). Then

$$b_t \cdot y = b_t \cdot (\varepsilon \cdot x) = b_t \varepsilon b_t^{-1} \cdot (b_t \cdot x) \qquad (\star)$$

Of course $b_t \varepsilon b_t^{-1}$ might not be the smallest displacement between $b_t \cdot y$ and $b_t \cdot x$, but if ε is small

[1] For the measure classification theorem it will suffice to take Γ discrete, or even closed, but we won't concern ourselves with this.

enough this calculation can be repeated several times. So it makes sense to study the conjugation action of our flow on G, or equivalently the adjoint representation $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$.

Recall we also have the adjoint action of the Lie algebra, which is related to Ad via log and exp, which are local diffeomorphisms. If Ad_{b_t} has only 1 as an eigenvalue, then $N = \log(I + (b_t - I))$ is nilpotent, and $\exp(N)$ is a polynomial. So we might hope the dynamics in this case will be particularly simple to understand in this setting.

Definition 1.1. An element $u \in G$ is *unipotent* if all the eigenvalues of Ad_u are $+1$ (if G is linear we simply ask that the eigenvalues of u are $+1$). A *unipotent flow* is a one-parameter subgroup of unipotent elements $U = \{u_s \mid s \in \mathbb{R}\}$.

We now recall from last time the following statements, collectively known as *Ratner's Theorems*, proven in [3, 4, 5]. In all that follows $X = \Gamma\backslash G$ where Γ is a lattice, and $U = \{u_s \mid s \in \mathbb{R}\}$ is a unipotent flow.

Theorem 1.2 (Dani's conjecture; Ratner's measure classification)**.** *Every U-invariant ergodic probability measure μ on X is **algebraic**.*

That is, there exists a closed, connected, unimodular subgroup $U \leq L \leq G$ such that μ is the L-invariant normalised probability measure (that is, the normalised Haar measure $m_{L\cdot x_0}$) on a closed orbit $L \cdot x_0$ (for any $x_0 \in \mathrm{supp}(\mu)$).

Theorem 1.3 (Ratner's equidistribution theorem)**.** *For any $x_0 \in X$ there exists a closed connected unimodular subgroup $U \leq L \leq G$ such that:*

(i) $L \cdot x_0$ *is closed with finite L-invariant volume;*

(ii) $\displaystyle \frac{1}{T}\int_0^T f(u_s \cdot x_0)\, ds \longrightarrow \frac{1}{\mathrm{vol}(L \cdot x_0)} \int_{L\cdot x_0} f \, dm_{L\cdot x_0}$ *as* $T \to \infty$.

Theorem 1.4 (Ranghunathan's conjecture; Ratner's orbit closure theorem)**.** *Let $H \leq G$ be a subgroup generated by unipotent flows. Then for any $x_0 \in X$ the orbit closure is **algebraic**.*

That is, there exists some closed connected unimodular group $H \leq L \leq G$ such that

$$\overline{H \cdot x_0} = L \cdot x_0$$

and $L \cdot x_0$ supports a finite L-invariant measure.

Our aim today is to prove a special case of the measure classification theorem.

2 First ideas in unipotent dynamics

We have our measure μ, and we want it to be invariant under some closed subgroup $L \leq G$. What should this L be?

A reasonable guess (really, the only guess) is to consider $\mathrm{Stab}(\mu)$, or even better $L := \mathrm{Stab}(\mu)^\circ$. Clearly $U \leq L$, L is closed, and any element close enough to the identity in $\mathrm{Stab}(\mu)$ is in fact in L. To see this last point recall that $\exp : \mathrm{Lie}(L) \to \mathrm{Stab}(\mu)$ is a local diffeomorphism whose image lies in (in fact generates) L.

We want to show that μ is supported on a single orbit of L, which we will achieve indirectly — if μ is not supported on a single orbit of a particular subgroup H which leaves the measure invariant, then we show that we can find arbitrarily small elements $\notin H$ that also stabilise μ. For L, this will lead to a contradiction.

In the setting of the measure classification theorem this is all we need, thanks to the following result:

Lemma 2.1. *Let $X = \Gamma\backslash G$ be a quotient of a Lie group G by a discrete subgroup Γ. Let H be a connected subgroup of G, and let μ be an H-invariant and ergodic probability measure. If μ is concentrated on a single orbit of* $\mathrm{Stab}(\mu)$*, then in fact μ is the Haar measure on a closed orbit of* $\mathrm{Stab}(\mu)^{\circ}$.

It is a general fact that finite volume orbits are closed, see for example [2, Proposition 1.22].

Observe now that if there is some $x_0 \in X$ and $\delta > 0$ such that $\mu(B_L^{\delta} \cdot x_0) > 0$, by ergodicity of μ we must have that $\mu(L \cdot x_0) = 1$, and by lemma 2.1 we would obtain the measure classification. Hence we will assume that $\mu(L \cdot x) = 0$ for all x.

Now we turn to obtaining the additional invariance we seek.

2.1 Centraliser gives invariance

In some very special cases the additional invariance comes almost for free.

Definition 2.2. $x \in X$ is *generic* with respect to μ and $U = \{u_s \mid s \in \mathbb{R}\}$ if

$$\frac{1}{T}\int_0^T f(u_s \cdot x)\,ds \longrightarrow \int_X f\,d\mu$$

as $T \to \infty$ for all $C_c(X)$.

The pointwise ergodic theorem and separability of $C_0(X)$ tell us that μ-a.e. $x \in X$ is generic if and only if μ is invariant and ergodic under U.

Lemma 2.3 (Centraliser lemma)**.** *If $x, y = h \cdot x$ are generic for μ and*

$$h \in C_G(U) := \{h \in G \mid gu = ug \text{ for all } u \in U\}$$

then h preserves μ.

Proof. We know that

$$\frac{1}{T}\int_0^T f(u_s \cdot y)\,ds \longrightarrow \int_X f\,d\mu$$

for any $f \in C_c(X)$ as y is generic. We can also calculate

$$\begin{aligned}\frac{1}{T}\int_0^T f(u_s \cdot y)\,ds &= \frac{1}{T}\int_0^T f(u_s \cdot (h \cdot x))\,ds\\ &= \frac{1}{T}\int_0^T f(h \cdot (u_s \cdot x))\,ds\\ &= \frac{1}{T}\int_0^T f \circ L_h(u_s \cdot x)\,ds\\ &\longrightarrow \int_X f \circ L_h\,d\mu\end{aligned}$$

So μ is h-invariant. □

2.2 Polynomial divergence leading to invariance

So far we haven't used the fact that U is unipotent, so recall the calculation ($\star$). In fact, $\mathrm{Ad}_{u(s)}$ is now a (matrix valued) polynomial in s.

Given nearby points x and $y = \varepsilon \cdot x$, let $v = \log \varepsilon$, and consider the $\mathfrak{g}$-valued polynomial $\mathrm{Ad}_{u(s)}(v)$. If ε is very small, this polynomial is close to zero in the space of all polynomials. However, we can speed up our flow by choosing a large T and considering the polynomial

$$p(r) = \mathrm{Ad}_{u(rT)}(v)$$

If our original polynomial is non-constant (equivalently, $\varepsilon \notin C_G(U)$), we can choose T so that the polynomial p belongs to a compact set of polynomials not containing the zero polynomial. In fact if $T > 0$ is the smallest number with $\|\mathrm{Ad}_{u(T)}(v)\| = 1$, then

$$\sup_{r \in [0,1]} \|p(r)\| = 1$$

Moreover, p is a polynomial of bounded degree.

Remark 2.4. This behaviour (that accelerating a polynomial is again a polynomial from the same finite dimensional space) is specific to polynomials, and hence to unipotent flows. By contrast, diagonalisable flows give exponential maps — accelerating these change the base of the exponential functions.

We will use this polynomial divergence to obtain more invariance, but first we need a uniform version of genericity.

Definition 2.5. A set $K \subset X$ is a set of *uniformly generic* points if for any $f \in C_c(X)$ and $\varepsilon > 0$ there is some $T = T_0(f, \varepsilon)$ with

$$\left| \frac{1}{T} \int_0^T f(u_s \cdot x)\, ds - \int_X f\, d\mu \right| < \varepsilon$$

for all $T \geq T_0$ and all $x \in K$.

Let us first see why his leads to more invariance, before finding large sets of uniformly generic points.

Proposition 2.6 (Polynomial divergence leads to invariance)**.** *Suppose that (x_n) and (y_n) are two sequences of uniformly generic points with*

(i) $x_n \to z$, $y_n \to z$;

(ii) $y_n = \varepsilon_n \cdot x_n$ with $\varepsilon_n \to I$ and $\varepsilon_n \notin C_G(U)$ for all $n \geq 1$.

Let $v_n := \log(\varepsilon_n)$ and consider the polynomials

$$p_n(r) := \mathrm{Ad}_{u(T_n r)}(v_n)$$

where we've picked the speeding-up parameter $T_n \to \infty$ such that

$$\sup_{r \in [0,1]} \|p_n(r)\| = 1$$

for each n. Suppose that $p_n(r) \to p(r)$ for all $r \in [0,1]$ where

$$p : \mathbb{R} \to \mathfrak{g}$$

is a polynomial with entries in $\mathfrak{g}$. Then μ is preserved by $\exp(p(r))$ for all $r \in \mathbb{R}_{\geq 0}$.

Remark 2.7. The assumption that $\varepsilon_n \notin C_G(U)$ is unproblematic, since if some $\varepsilon_n \in C_G(U)$ we might be able to apply the centraliser lemma. Similarly the assumption that the polynomials converge is mild as they lie in a compact subset of a finite-dimensional space, so we may pass to a convergent subsequence.

Proof. Fix some $r_0 \in \mathbb{R}_{>0}$, $f \in C_c(X)$, and $\varepsilon > 0$. Since f is uniformly continuous there is some $\delta = \delta(f, \varepsilon) > 0$ with

$$d(h_1, h_2) < \delta \quad \Longrightarrow \quad |f(h_1 \cdot x) - f(h_2 \cdot x)| < \varepsilon$$

for all $x \in X$. Furthermore choose $\kappa > 0$ such that

$$d(\exp p(r), \exp p(r_0)) < \delta/2$$

for all $r \in [r_0 - \kappa, r_0]$. Then there is an N such that we also have

$$d(\exp p_n(r), \exp p(r_0)) < \delta \tag{$\star\star$}$$

for all $n \geq N$ and $r \in [r_0 - \kappa, r_0]$.

Since the x_n are uniformly generic we have that

$$\frac{1}{r_0 T_n} \int_0^{r_0 T_n} f(u_s \cdot x_n)\, ds \longrightarrow \int_X f\, d\mu$$

and

$$\frac{1}{(r_0 - \kappa) T_n} \int_0^{(r_0 - \kappa) T_n} f(u_s \cdot x_n)\, ds \longrightarrow \int_X f\, d\mu$$

(both as $n \to \infty$). Taking the correct linear combination (with $\kappa > 0$ fixed) and replacing f by $f \circ L_{\exp(p(r_0))}$ we get

$$\frac{1}{\kappa T_n} \int_{(r_0 - \kappa) T_n}^{r_0 T_n} f \circ L_{\exp(p(r_0))}(u_s \cdot x_n)\, ds \longrightarrow \int_X f \circ L_{\exp(p(r_0))}\, d\mu.$$

By the same argument we also have

$$\frac{1}{\kappa T_n} \int_{(r_0 - \kappa) T_n}^{r_0 T_n} f(u_s \cdot y_n)\, ds \longrightarrow \int_X f\, d\mu.$$

Using our definition of v_n and p_n we have that

$$\begin{aligned} u_s \cdot y_n &= u_s \exp(v_n) \cdot x_n \\ &= \exp(\mathrm{Ad}_{u_s}(v_n))(u_s \cdot x_n) \\ &= \exp(p_n(s/T_n))(u_s \cdot x_n) \end{aligned}$$

for all $s \in \mathbb{R}$.

Restricting now to $s \in \mathbb{R}$ with $\frac{s}{T_n} \in [r_0 - \kappa, r_0]$, together with ($\star\star$), we deduce that

$$d(u_s \cdot y_n, \exp p(r_0) u_s \cdot x_n) < \delta$$

and so

$$|f(u_s \cdot y_n) - f(\exp p(r_0) u_s \cdot x_n)| < \varepsilon$$

for every $s \in [(r_0 - \kappa)T_n, r_0 T_n]$. Using this estimate in the integrals above gives

$$\left| \frac{1}{\kappa T_n} \int_{(r_0-\kappa)T_n}^{r_0 T_n} f \circ L_{\exp(p(r_0))}(u_s \cdot x_n)\, ds - \frac{1}{\kappa T_n} \int_{(r_0-\kappa)T_n}^{r_0 T_n} f(u_s \cdot y_n)\, ds \right| < \varepsilon$$

and so

$$\left| \int_X f \circ L_{\exp p(r_0)}\, d\mu - \int_X f\, d\mu \right| \leq \varepsilon$$

Since this holds for any $\varepsilon > 0$ and $f \in C_c(X)$ we have invariance of μ under $\exp p(r_0)$. Since $r_0 > 0$ was arbitrary, the result follows. □

2.3 Large sets of uniformly generic points

Now back to the problem of finding large sets of uniformly generic points.

Lemma 2.8 (Large sets of uniformly generic points)**.** *Let μ be an invariant and ergodic probability measure on X for the action of a one-parameter flow $\{u_s \mid s \in \mathbb{R}\}$. For any $\rho > 0$ there is a compact $K \subset X$ with $\mu(K) > 1 - \rho$ consisting of uniformly generic points.*

Proof. Let $D = \{f_1, f_2, \dots\} \subset C_c(X)$ be countable and dense. Then by the pointwise ergodic theorem for every $f_\ell \in D$ we have

$$\frac{1}{T} \int_0^T f_\ell(u_s \cdot x)\, ds \longrightarrow \int_X f_\ell\, d\mu$$

for μ-a.e. x.

Equivalently for every $\varepsilon > 0$ we have

$$\mu\left(\left\{ x \in X \;\middle|\; \sup_{T > T_0} \left| \frac{1}{T} \int_0^T f_\ell(u_s \cdot x)\, ds - \int_X f_\ell\, d\mu \right| > \varepsilon \right\} \right) \longrightarrow 0$$

as $T_0 \to \infty$. Now choose for every $f_\ell \in D$ and for every $\varepsilon = \frac{1}{n}$ a time $T_{\ell,n}$ so that

$$\mu\left(\left\{ x \in X \;\middle|\; \sup_{T > T_{\ell,n}} \left| \frac{1}{T} \int_0^T f_\ell(u_s \cdot x)\, ds - \int_X f_\ell\, d\mu \right| > \frac{1}{n} \right\} \right) < \frac{\rho}{2^{\ell+n}}$$

Let $K' \subset X$ be the complement of the union of these sets, so that $\mu(K') > 1 - \rho$ by construction. It is clear that the points of K' are uniformly generic for all $f \in D$, and by density of $D \subset C_c(X)$ in the uniform norm this extends to all functions. Finally, we may choose a compact $K \subset K'$ with $\mu(K) < 1 - \rho$, by regularity of μ. □

3 Semisimple groups

We are studying the adjoint representation, and so we might hope to deduce invariance when the representation theory of our acting group has good properties. In particular, as noticed by Einsiedler in [1], in the case our acting group is semisimple with no compact factors the proof simplifies. Here, the Mautner phenomenon (as explained by Konstantin two weeks ago) can be used to find an ergodic unipotent flow.

Theorem 3.1 (Ratner measure classification; semisimple case, [1]). *Let G be a connected Lie group, $\Gamma \leq G$ discrete, and $H \leq G$ semisimple without compact factors. Suppose that μ is an H-invariant and ergodic probability measure on X. Then μ is **algebraic**.*

As indicated before, set $L := \mathrm{Stab}(\mu)^\circ$ with Lie algebra $\mathfrak{l}$. We need to show that μ is concentrated on a single L-orbit (and then use Lemma 2.1), let us assume this is not the case. By ergodicity of μ, every L-orbit must have measure 0 since $H \leq L$.

Claim 1 (Reduce to $\mathrm{SL}(2,\mathbb{R})$ case). *H contains a subgroup that is locally isomorphic to $\mathrm{SL}(2,\mathbb{R})$ which acts ergodically on X with respect to μ.*

Proof. By assumption H is an almost direct product of simple Lie groups, and each of these contains a subgroup that is locally isomorphic to $\mathrm{SL}(2,\mathbb{R})$, so consider the diagonally embedded group. It projects non-trivially to each factor, and by the Mautner phenomenon for H acts ergodically. □

So we can assume that H itself is locally isomorphic to $\mathrm{SL}(2,\mathbb{R})$. Let $U \leq \mathrm{SL}(2,\mathbb{R})$ be the upper unipotent group, by the Mautner phenomenon again U acts ergodically with respect to μ.

Since $\mathrm{SL}(2,\mathbb{R})$ is completely reducible, the H-invariant subspace $\mathfrak{l} \leq \mathfrak{g}$ (with the adjoint action) has an H-invariant complement $V \leq \mathfrak{g}$.

Now let $K \subset X$ be a set with $\mu(K) > 0.99$ consisting of uniformly generic points for $U \leq H$ (by lemma 2.8).

Claim 2. There are points $x_n, y_n \in K$ with $y_n = g_n \cdot x_n$ such that

(i) $g_n \neq I$, $g_n \to I$ as $n \to \infty$;

(ii) $g_n \in \exp(V)$ (that is, the g_n belong in the transverse direction to L).

We will consider the polynomials

$$p_n(r) = \mathrm{Ad}_{u(T_n r)}(\log g_n) \tag{†}$$

(or a convergent subsequence of these) and apply Proposition 2.6. Since V is H-invariant all these polynomials, and hence their limit $p : \mathbb{R} \to \mathfrak{g}$ takes values in V. Hence μ is preserved by $\exp p(r)$ for all $r > 0$, which contradicts the definition of $L = \mathrm{Stab}(\mu)^\circ$.

Let B^L_δ be a small open ball in L around the identity, and define

$$Y = \left\{ x \in X \,\middle|\, \int_{B^L_\delta} \chi_K(\ell \cdot x)\, dm_L(\ell) > 0.9\, m_L(B^L_\delta) \right\}$$

Claim 3. $\mu(Y) > 0.9$.

Proof. We calculate

$$\begin{aligned}
\mu(X \backslash Y) &= \mu\left(\left\{ x \in X \,\middle|\, \int_{B^L_\delta} \chi_{X\backslash K}(\ell \cdot x)\, dm_L(\ell) \geq 0.1\, m_L(B^L_\delta) \right\} \right) \\
&\leq \frac{1}{0.1\, m_L(B^L_\delta)} \int_X \int_{B^L_\delta} \chi_{X\backslash K}(\ell \cdot x)\, dm_L(\ell)\, d\mu \\
&= \frac{1}{0.1\, m_L(B^L_\delta)} \int_{B^L_\delta} \int_X \chi_{X\backslash K}(\ell \cdot x)\, d\mu\, dm_L(\ell) \\
&= \frac{\mu(X\backslash K)}{0.1} < \frac{0.01}{0.1} = 0.1
\end{aligned}$$

and hence $\mu(Y) > 0.9$. ☐

Claim 4. For any nearby points $x, y \in Y$ we can find $\ell_x, \ell_y \in B^L_\delta$ such that

(P1) $x' = \ell_x \cdot x \in K$;

(P2) $y' = \ell_y \cdot y \in K$;

(P3) $y' = \exp(v) \cdot x'$ with $v \in V$.

Proof. By definition of Y, at least 90% of all $\ell_x \in B^L_\delta$ satisfy (P1), and similarly 90% of $\ell_y \in B^L_\delta$ satisfy (P2). However, we would like to do this while ensuring (P3) holds.

If δ is sufficiently small then

$$\begin{aligned} \varphi : B^L_{2\delta} \times B^V_{2\delta}(0) &\to G \\ (\ell, v) &\mapsto \ell \exp v \end{aligned}$$

is a diffeomorphism onto an open neighbourhood $I \in \mathcal{O} \subset G$ (using the inverse mapping theorem).

Pick $g \in B^G_\kappa$ chosen with $y = g \cdot x$, then we want to find $\ell_x, \ell_y \in B^L_\delta$ with $g\ell_x^{-1} = \ell_y^{-1} \exp v$, which would give (P3). Indeed, using the local diffeomorphism above, if κ is sufficently small then $g\ell_x^{-1} \in \mathcal{O}$ and we can define

$$(\ell_y, v) = \varphi^{-1}(g\ell_x^{-1}) \tag{‡}$$

Define the map

$$\phi : B^L_\delta \to B^L_{2\delta} : \ell_x \mapsto \ell_y$$

with ℓ_y as in (‡). This is a smooth map that depends on the parameter $g \in B^G_\kappa$, and for κ sufficiently small is close to the identity in the C^1-topology. Therefore ϕ doesn't distort thw chosen Haar measure of L much, and sends B^L_δ into a ball around the identity that isn't much bigger than B^L_δ.

In other words, for κ sufficiently small,

$$\begin{aligned} m_L\big(\phi(\{\ell_x \in B^L_\delta \mid \ell_x \cdot x \in K\}) \cap B^L_\delta\big) &> 0.9\, m_L\big(\phi(\{\ell_x \in B^L_\delta \mid \ell_x \cdot x \in K\})\big) \\ &> 0.8\, m_L\big(\{\ell_x \in B^L_\delta \mid \ell_x \cdot x \in K\}\big) \\ &> (0.8)\,(0.9) m_L(B^L_\delta) > 0.7 m_L(B^L_\delta) \end{aligned}$$

Together with

$$m_L\big(\phi(\{\ell_x \in B^L_\delta \mid \ell_x \cdot x \in K\})\big) > 0.9\, m_L(B^L_\delta)$$

we see that there are many points $\ell_x \in B^L_\delta$ satisfying (P1) such that ℓ_y defined using (‡) also satisfies (P2). ☐

Proof of claim 2. Let $z \in \operatorname{Stab}(\mu) \cap Y$. Then for every $\kappa = \frac{1}{n}$ there exist $x_n = z$, $y_n = g_n \cdot x_n \in Y$ with

$$g_n \in B^G_{1/n} \backslash L$$

Applying claim 4 above to x_n, y_n for n large enough we get

$$x_n',\ y_n' = \exp v_n \cdot x_n' \in K,\ v_n \in V,\ v_n \neq 0,\ v_n \to 0$$

☐

Proof of Theorem. There are two cases to consider

- If v_n is in the eigenspace of Ad_{u_s} for infinitely many n (let's assume for all n) then apply the centraliser lemma to deduce that $\exp(v_n)$ preserves μ.

  By passing again to a subsequence, we may assume that $\frac{v_n}{\|v_n\|} \to w$ in the unit sphere of V. Since $\mathrm{Stab}(\mu)$ is closed, $\exp(tw) \in \mathrm{Stab}(\mu)$ for all t. But since V is a linear complement to the Lie algebra of $L = \mathrm{Stab}(\mu)^\circ$, this is a contradiction.

- Assume v_n is not in an eigenspace for any $n \geq 1$ (delete finitely many terms), define T_n so that the polynomials in † have norm one. By compactness a sequence converges to p say, and by Proposition 2.6 μ is preserved by $\exp p(t)$ for all $t > 0$. p takes values in V, again contradicting the definition of V.

□

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