List of Problems - 1^{st} week

1. **a)** Prove Poincare's Recurrence for a measure preserving transformation $T : X \to X$ on a probability space (X, μ): Let A be a Borel-measurable set of positive measure. Show that for μ-a.e. $x \in A$ there exist infinitely many $n > 0$ such that $T^n x \in A$.

Hint: Study the set $\{x \in X : \forall\, n > 0 \quad T^n x \notin A\}$.

b) Use the above result to get the following topological version: Assume also that X is a separable metric space. Prove that there exists a sequence $\{n_k\}$ such that $T^{n_k} x \to x$ for μ-a.e. $x \in X$.

2. Prove that for a σ-compact locally compact metrizable group G there exists a left Haar measure.

3. Prove that the Haar measure is unique up to scalar multiple.

4. Let G be a connected σ-compact locally compact group G equipped with a proper left-invariant metric. Let Γ be a lattice in G such that the quotient space $\Gamma\backslash G$ is compact. Show that Γ is finitely generated in the sense that there exists a finite set $S \subset \Gamma$ so that any element of Γ is a product of elements in S.

Hint: Show that if B is a ball in G containing an open neighborhood of a fundamental domain of Γ then $S = \{\gamma \in \Gamma : \gamma B \cap B \neq \emptyset\}$ is a generating set.

5. Show that $\mathrm{SL}_2(\mathbb{R})$ does not have a bi-invariant metric and conclude the same for $\mathrm{SL}_n(\mathbb{R})$, $n > 2$.

Hint: Consider the conjugation action of a diagonal element (say $\operatorname{diag}(e^t, e^{-t})$) to the unipotent elements $\left[\begin{smallmatrix} 1 & s \\ 0 & 1 \end{smallmatrix}\right]$ and their transposes. The latter generate $\mathrm{SL}_2(\mathbb{R})$...

6. Give an explicit construction of the Haar measure of a group G if it is also a differentiable manifold (e.g. if G is a Lie group). Use it to describe the Haar measure of

$$B = \left\{ \begin{bmatrix} a & b \\ & d \end{bmatrix} : \; a, b, d \in \mathbb{R} \text{ and } ad = 1 \right\} < \mathrm{SL}_2(\mathbb{R})$$

7. Let G be a unimodular Lie group satisfying $G = ST$ for two closed subgroups S and T of G for which $S \cap T = \{e\}$. Then the invariant measure on G is given by

$$\int_G f\, dm = \int_{S \times T} f(st^{-1})\, dm_S(s)\, dm_T(t), \quad f \in C_c(G),$$

where m_S and m_T are the left-invariant measures on S and T respectively.

Hint: Show that $(\varphi^{-1})_* m$ is a Haar measure on $S \times T$ where $\varphi(s, t) = st^{-1}$.

Remark: Exercises 2., 3. and 6. are supposed to be solely used as presentation problems for which you may contact rene.ruehr@math.ethz.ch for references.