2^{nd} List of Problems

1. Characterize the periodic orbits of $U^+ = \{\left[\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right]\}$ in $\Gamma\backslash \mathrm{SL}_2(\mathbb{R})$ where $\Gamma = \mathrm{SL}_2(\mathbb{Z})$!

a) Show that the the orbit at Γg is periodic if and only if $a/c \in \mathbb{Q} \cup \infty$ where $g = \left[\begin{smallmatrix} a & * \\ c & * \end{smallmatrix}\right]$.

b) If Γg has a periodic orbit then any other orbit is of the form $\Gamma g U a$ for some diagonal $a \in A$.

2. Prove the following warm-up version of the Anosov Shadowing Lemma of $X = \Gamma\backslash \mathrm{SL}_2(\mathbb{R})$: For nearby $x, y \in X$ there exists another point z such that $a_t \cdot z \approx a_t \cdot x$ in forward time ($t \geq 0$) and $a_t \cdot z \approx a_t \cdot y$ in backward time ($t \leq 0$).

Hint: Use the fact that any small enough $g \in G$ can be written as a product of elements in A, U^+ and $U^- = \{\left[\begin{smallmatrix} 1 & 0 \\ * & 1 \end{smallmatrix}\right]\}$.

3. Prove the Anosov Closing Lemma: For any ε and any time T there exists $\delta > 0$ such that if $x \in X = \Gamma\backslash \mathrm{SL}_2(\mathbb{R})$ is δ-close to $a_T \cdot x$ then there exists $z \in X$, ε-close to x and a time S also ε-close to T so that $a_S \cdot z = z$.

Hint: Decompose the difference between x and $a_T \cdot x$ as above. Kill the diagonal direction by changing T to S. Then get rid of the U^--part using the fact that conjugating with a_S defines a contraction for which you can use Banach's fixpoint theorem. Repeat this step for U^+.

4. Prove that any $g \in \mathrm{SL}_2(\mathbb{R})$ is conjugate (up to sign) to an element of A, U^- or $\mathrm{SO}(2, \mathbb{R})$.

5. (Presentation) Prove the pointwise ergodic theorem!

6. (Presentation) Give the key ideas of Hopf's argument for ergodicity of the geodesic flow with respect to the Haar measure on $\Gamma\backslash \mathrm{SL}_2(\mathbb{R})$.

7. Prove that U^+ and A are unimodular and describe their Haar measures.

8. Let λ_i, $i = 1 \ldots d$, be the successive minima of a lattice Λ in $\mathbb{R}^d$. Mahler's compactness criterion is phrased by giving a lower bound for λ_1. Can it be reformulated in terms of λ_i for $i > 1$?