3^{rd} List of Problems

1. (Presentation) Prove Hilbert's Nullstellensatz: For an algebraically closed field K and a variety $Z(I)$ it holds that

$$f \in I(Z(I)) \text{ if and only if } f \in \operatorname{Rad}(I)$$

with functions I, Z and Rad defined as in Exercise 2.

2. Check the following algebraic-geometric dictionary which we used implicitly in class by applying the Nullstellensatz!

Algebra		**Geometry**
$I(Z)$	$\longrightarrow$	$Z(I(Z)) = Z$
$I(Z(I)) = \operatorname{Rad}(I)$	$\longleftarrow$	$Z(I)$
$I + J$	$\longrightarrow$	$Z(I) \cap Z(J)$
$\operatorname{Rad}(I(Z) + I(Y))$	$\longleftarrow$	$Z \cap Y$
IJ	$\longrightarrow$	$Z(I) \cup Z(J)$
$\operatorname{Rad}(I(Z)I(Y))$	$\longleftarrow$	$Z \cup Y$
$I \cap J$	$\longrightarrow$	$Z(I) \cup Z(J)$
prime ideal		connected variety
maximal ideal		point of affine space
ascending chain condition		descending chain condition

where I, J are ideals in $K[x_1, \ldots, x_d]$ and Z, Y are varieties in K^d over an algebraically closed field K and

$$Z(I) = \{x \in K^d : f(x) = 0 \text{ for all } f \in I\},$$

$$I(Z) = \{f \in K[x_1, \ldots, x_d] : f(x) = 0 \text{ for all } x \in V\},$$

$$\operatorname{Rad}(I) = \{f \in K[x_1, \ldots, x_d] : f^m \in I \text{ for some } m > 0\}.$$

3. Let $H < \mathrm{SL}_d$ be an unipotent algebraic group defined over $\mathbb{Q}$.

a) Show that $H(\mathbb{R})$ has a closed orbit in $\mathrm{SL}_d(\mathbb{Z})\backslash \mathrm{SL}_d(\mathbb{R})$

Hint: Use Chevalley's theorem and the stablizer subgroup proposition.

b) Show that $H(\mathbb{R})$ has a compact orbit in $\mathrm{SL}_d(\mathbb{Z})\backslash \mathrm{SL}_d(\mathbb{R})$

Hint: Treat first the case of subgroups of the upper triangular subgroup, then discuss what happens when you conjugate with a rational matrix.