List of Problems 4

This problem set asks you to provide details of the following facts that appeared in the course. Let as usual X_2 denote the space $\mathrm{SL}_2(\mathbb{Z})\backslash \mathrm{SL}_2(\mathbb{R})$.

1. Apply the quantitative non-divergence theorem to prove that any limit of a sequence of measures of the form $\frac{1}{T_n}\int_0^{T_n} \delta_{xu_t} dt$ is a probability measure for $x \in X_2$ where u_t denotes the horocycle flow.

2. Use the measure classification for the horocycle flow on X_2 to show that any U-invariant measure that gives zero mass to the set of all periodic orbits of the horocycle flow must be the Haar measure of X.

3. Prove that if an ergodic measure of the horocycle flow on X_2 has the property that for any two generic points x and y there exists t such that $x = yu_t$ then it must be supported on a compact orbit.

4. Let $x \in X_2$ give rise to a periodic orbit of the horocycle flow and let μ be the normalized Lebesgue measure of that orbit. Prove that the measures $(a_t)_*\mu$ diverge to the trivial measure and equidistribute with respect to the Haar measure as $t \to \infty$ and $t \to -\infty$ respectively.

5. Show that the limit polynomial of the sequence $p_n(r) = Ad_{u(T_n r)}(v_n)$ as defined in the lecture only takes values in the centralizer $\{v \in \mathfrak{g} : \mathrm{Ad}_u(v) = v \ \forall u \in U\}$ of U in the Lie algebra $\mathfrak{g}$ of G.

6. Prove that the flow $u_t : \mathbb{T}^2 \to \mathbb{T}^2$ on the 2-torus given by $x \mapsto x + (a, b)t$ is uniquely ergodic if a and b are independent over $\mathbb{Q}$.