D-MATH Diophantine Geometry FS 2013
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Exercise Sheet 1

1. **a)** Show that (2) is a maximal ideal in the ring $\mathbb{Z}[\frac{\sqrt{5}+1}{2}]$. What is its residue field, i.e. $\mathbb{Z}[\frac{\sqrt{5}+1}{2}]/(2) = ?$

b) Let $\mathcal{O} = \mathbb{Z}[\sqrt{-5}]$, and let us take the ideals $\mathfrak{p} = (2, 1+\sqrt{-5})$, $\mathfrak{p}' = (2, 1-\sqrt{-5})$, $\mathfrak{q} = (3, 1+\sqrt{-5})$. Show that these are maximal ideals of $\mathcal{O}$, and calculate the residue fields $\mathcal{O}/\mathfrak{p}$, $\mathcal{O}/\mathfrak{p}'$, $\mathcal{O}/\mathfrak{q}$.

c) Show that 2 and $1+\sqrt{-5}$ are irreducible elements of $\mathcal{O}$. Furthermore, show that $(2) = \mathfrak{p}\mathfrak{p}'$ and $(1+\sqrt{-5}) = \mathfrak{p}\mathfrak{q}$. This demonstrates that the ideal generated by an irreducible element is not necessarily a prime ideal.

d) Show that $\mathcal{O}$ is not a unique factorization domain. (A unique factorization domain is an entire ring in which every element a can be written as a product $a = \varepsilon p_1 \dots p_n$, where ε is a unit and the p_i are irreducible elements, and this factorization is unique up to reordering and multiplying by units.)

e) Show that a unique factorization domain is integrally closed in its fraction field.

2. Show that $\mathcal{B} = (1, \sqrt[3]{2}, \sqrt[3]{2}^2)$ is an integral basis in the number field $K = \mathbb{Q}(\sqrt[3]{2})$. Calculate the discriminant of K using the definition from the lecture. Compare the result with the discriminant of the polynomial $X^3 - 2$.

3. Let A be an integrally closed entire ring, and $K = \mathrm{Frac}(A)$ the fraction field of A. Let L/K be a finite field extension, and let $\alpha \in L$. The minimal polynomial of α over K is $P(X) = X^n + a_1X^{n-1} + \ldots + a_n \in K[X]$.

a) Show that α is integral over A if and only if $P \in A[X]$.

b) If α is integral over A, then $\mathrm{Nm}_{L/K}(\alpha), \mathrm{Tr}_{L/K}(\alpha) \in A$.

c) Let B be the integral closure of A in L. Show that α is a unit in B if and only if $\mathrm{Nm}_{L/K}(\alpha)$ is a unit in A.

Bitte wenden!

4. Let K be a number field with integral basis $\mathcal{B} = \{\omega_1, \dots, \omega_n\}$.

a) Show that

$$\beta : K \times K \to \mathbb{Q}, \quad (\omega, \omega') \mapsto \mathrm{Tr}_{K/\mathbb{Q}}(\omega\omega')$$

is a symmetric, non-degenerate $\mathbb{Q}$-bilinear map. (Non-degenerateness means that if $\omega \in K$ is such that $\beta(\omega, \omega') = 0$ for every $\omega' \in K$, then $\omega = 0$.)

b) Show that the discriminant of K is $\Delta_K = \det(\mathrm{Tr}_{K/\mathbb{Q}}(\omega_i\omega_j))_{1\leq i,j\leq n}$. Using this definition for the discriminant, deduce that $\Delta_K \in \mathbb{Z}$, and also show that $\Delta_K \neq 0$.