Exercise Sheet 3

Notation: If K is a number field and $x \in K$, then $h_K(x) := \sum_v \log^+(\|x\|_v)$.

If $x \in \overline{\mathbb{Q}}$, then $h(x) := \frac{1}{[K:\mathbb{Q}]} h_K(x)$ is absolute logarithmic height of x, where K is any number field containing x.

Let K be a number field and let $x_1, \ldots, x_n \in K$, and $x = (x_1, \ldots, x_n) \in \mathbb{A}^n_K$. Let v be a place of K. We define two absolute values:

$$\|x\|_v := \begin{cases} \max\{\|x_i\|_v;\ 1 \le i \le n\} & v \text{ non-archimedean,} \\ (\sum_{i=1}^n \tau(x_i)^2)^{1/2} & v \text{ real,} \\ \sum_{i=1}^n \tau(x_i)\overline{\tau(x_i)} & v \text{ complex,} \end{cases}$$

and

$$\|x\|'_v := \max\{\|x_i\|_v;\ 1 \le i \le n\}.$$

If K is a number field and $x = (x_0 : \ldots : x_n) \in \mathbb{P}^n_K$, then

$$h_K(x) := \sum_v \log \|x\|_v \quad \text{and} \quad h'_K(x) := \sum_v \log \|x\|'_v.$$

If $x = (x_0 : \ldots : x_n) \in \mathbb{P}^n_{\overline{\mathbb{Q}}}$, then

$$h(x) := \frac{1}{[K:\mathbb{Q}]} h_K(x) \quad \text{and} \quad h'(x) := \frac{1}{[K:\mathbb{Q}]} h'_K(x),$$

where K is any number field containing $x_0, \ldots, x_n$.

If $x = (x_1, \ldots, x_n) \in \mathbb{A}^n_{\overline{\mathbb{Q}}}$, then

$$h(x) := h(x_1 : \ldots : x_n : 1) \quad \text{and} \quad h'(x) := h(x_1 : \ldots : x_n : 1).$$

Bitte wenden!

1. **a)** Let $P \in \mathbb{P}^n_{\overline{\mathbb{Q}}}$. Check that $0 \leq h'(P) \leq h(P) \leq h'(P) + \frac{1}{2}\log(n+1)$.

Hint: First show that if K is a number field, $x \in \mathbb{A}_K^{n+1}$, and v is an archimedean place of K, then $\|x\|'_v \leq \|x\|_v \leq (n+1)^{n_v/2}\|x\|'_v$ (here $n_v = 1$ for real places and $n_v = 2$ for complex places). Then use $[K : \mathbb{Q}] = \sum_{v|\infty} n_v$.

b) Check that if $x \in \overline{\mathbb{Q}}$, then $h(x) = h'(x : 1)$, where $(x : 1) \in \mathbb{P}^1_{\overline{\mathbb{Q}}}$.

c) Show that $h(xy) \leq h(x) + h(y)$ for every $x, y \in \overline{\mathbb{Q}}$.

d) Show that $h(x_1 + \cdots + x_m) \leq h(x_1) + \cdots + h(x_m) + \log(m)$ for every $x_1, \ldots, x_m \in \overline{\mathbb{Q}}$ and every $m \geq 1$.

Hint: First show that $\log^+(\sum_{i=1}^m a_i) \leq \log(m) + \sum_{i=1}^m \log^+(a_i)$ for $a_1, \ldots, a_m \in \mathbb{R}_{\geq 0}$ and $m \geq 1$.

e) More generally, show that $h'(P_1 + \cdots + P_m) \leq h'(P_1) + \cdots + h'(P_m) + \log(m)$ for every $P_1, \ldots, P_m \in \mathbb{A}^n_{\overline{\mathbb{Q}}}$.

2. Let $x \in \overline{\mathbb{Q}}$. Show that $h(x) = 0$ if and only if x is a root of unity. (This is called Kronecker's theorem).

Hint: To prove "$\Rightarrow$", take a Galois extension $K/\mathbb{Q}$ such that $x \in K$. Let $\sigma_1, \ldots, \sigma_n$ be the complex embeddings of K into $\mathbb{C}$, and consider the polynomial $f(T) = \prod_{j=1}^n (T - \sigma_j(x))$. Show that $f \in \mathbb{Z}[X]$ and the coefficients of f are bounded. Show that the same works for any power of x, and conclude that $x^i = x^j$ for some $i \neq j$.

3. We have used in Lecture 5 that $|f(z)| \leq c^{hk+L|z|}$ for $m_0 + \cdots + m_{n-1} < k$. Prove this. (For notation, see the notes.)

4. We have used in Lecture 5 that $|f(l) - g(l)| \leq B^{-\frac{1}{2}C}$ for $0 \leq l < hk^{\frac{1}{2(n+1)}}(\log k)^{-1}$ provided C is sufficiently large. Prove this. (For notation, see the notes.)