# Lie Groups II

## Exercise sheet 1

Each problem values one point (if one problem has $n$ sections, then the value of each section is $\frac{1}{n}$).

**Problem 1.**
1.-Define the tensor product of two Lie groups representations .
2.-If $G$ is a Lie group, $\rho : G \to \mathrm{GL}(V), \tau :\to \mathrm{GL}(W)$ are Lie group representations and $\mathrm{Lie}(\rho) : \mathrm{Lie}(G) \to \mathfrak{gl}(V)$ and $\mathrm{Lie}(\tau) : \mathrm{Lie}(G) \to \mathfrak{gl}(W)$ are their respectively associated Lie algebra representations, show that with the definition given above

$$\mathrm{Lie}(\rho \otimes \tau) \cong \mathrm{Lie}(\rho) \otimes \mathrm{Lie}(\tau).$$

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**Problem 2.** Let $G$ be a complex Lie group.
1.-Let $\rho : G \to \mathrm{GL}(V)$ be an irreducible representation. Show that if $Z(G)$ is the center of $G$, then there is $\tau : Z(G) \to \mathbb{C}^*$ continuous map such that

$$\rho(g) = \tau(g) Id_V \qquad \text{for all } g \in Z(G).$$

2.-What is the analogue for Lie algebras?

**Problem 3.** Let $V_n$ be the space of homogeneous polynomials of degree n in $\mathbb{C}[x, y]$ and take the following map

$$\rho_n : SL_2(\mathbb{C}) \longrightarrow \mathrm{GL}(V_n), \qquad \rho_n \begin{pmatrix} a & b \\ c & d \end{pmatrix} (q(x, y)) = q(ax + cy, bx + dy).$$

Show that this is an irreducible Lie group representation of dimension $n + 1$.

**Problem 4.** With the notation of the previous problem, show that

$$\rho_n \otimes \rho_m \cong \bigoplus_{j=0}^{\min\{m,n\}} \rho_{m+n-2j}.$$

Hint: Recall the combinatorial identity

$$\left(\sum_{j=0}^{k} x^{k-2j}\right)\left(\sum_{j=0}^{l} x^{l-2j}\right) = \sum_{j=0}^{\min\{k,l\}} \left(\sum_{s=0}^{k+l-2j} x^{k+l-2j-2s}\right)$$