Exercise Sheet 1

1. Fix $0 < \alpha < 1$. The aim of this exercise is to show that

$$S_\alpha(X) = \sum_{\substack{n \leq X \\ P_-(n) > X^\alpha}} 1 \sim c(\alpha) \frac{X}{\log X} \quad \text{as } X \to \infty, \tag{1}$$

where $P_-(n)$ denotes the smallest prime divisor of n.

1. Define $I_n = \left]\frac{1}{n+1}, \frac{1}{n}\right]$ and $J_n(X) = \left]X^{\frac{1}{n+1}}, X^{\frac{1}{n}}\right]$. Show that there exists $\beta(p)$ such that $\beta(p) \in I_n$ whenever $p \in J_{n+1}(X)$ and such that

$$S_\alpha(X) = \sum_{X^\alpha < p \leq X} 1 + \sum_{X^\alpha < p \leq X^{\frac{1}{2}}} S_{\beta(p)}\left(\frac{X}{p}\right).$$

2. Show that for $\alpha \in I_1$,

$$S_\alpha(X) \sim \frac{X}{\log X} \quad \text{as } X \to \infty.$$

For the subsequent steps, it actually turns out that we need the stronger statement

$$S_\alpha(X) = \frac{X}{\log X} - \frac{X^\alpha}{\alpha \log X} + O\left(\frac{X}{(\log X)^2}\right).$$

Prove it!

3. We prove (1) by induction. It is better, not to work with "$\sim$", but instead with an error term of the form $O\left(\frac{X}{(\log X)^2}\right)$. Suppose (1) holds for all $\alpha \in I_k$, $k < n$. Use two times integration by parts, to show that for all $\alpha \in I_n$

$$S_\alpha(X) = c(\alpha)\frac{X}{\log X} + \text{some error term.}$$

Hint: (i) Note that $c(\alpha)$ is bounded and continuously differentiable on any of the intervals I_k. (ii) After the first integration by parts, separate the expression you get into a main term and an error term, estimate the error term and apply integration by parts a second time to the main term.

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2. Fix a positive integer k. The aim of this exercise is to show that

$$\tau_k(X) = \sum_{\substack{n \le X \\ \Omega(n)=k}} 1 \sim \frac{1}{(k-1)!}\frac{X}{\log X}\big(\log(\log(X))\big)^{k-1} \quad \text{as } X \to \infty, \tag{2}$$

where $\Omega(n)$ denotes the number of prime factors of n counted with multiplicities.

1. Define
$$\Pi_k(X) = \sum_{\substack{(p_1,\dots,p_k) \\ p_1p_2\dots p_k \le X}} 1,$$
where the sum is over all ordered k-tuples of primes $(p_1, \dots, p_k)$ such that $p_1p_2\dots p_k \le X$. First, prove that
$$\Pi_k(X) \sim k\frac{X}{\log X}\big(\log(\log(X))\big)^{k-1} \tag{3}$$
implies (2). To do so, let
$$\pi_k(X) = \sum_{\substack{n \le X \\ n \text{ squarefree} \\ \Omega(n)=k}} 1$$
and show that
$$k!\pi_k(X) \le \Pi_k(X) \le k!\tau_k(X) \quad (k \ge 1) \tag{4}$$
as well as
$$\tau_k(X) - \pi_k(X) \le \Pi_{k-1}(X) \quad (k \ge 2). \tag{5}$$
Now, deduce (2) by using (3) in (4) and (5).
2. It remains to prove (3). Prove (3) for $k = 1$ (by using the prime number theorem). Define
$$\vartheta_k(X) = \sum_{\substack{(p_1,\dots,p_k) \\ p_1p_2\dots p_k \le X}} \log(p_1p_2\dots p_k).$$
By applying summation by parts to $\vartheta_k(X)$ as well as using the fact that $\Pi_k(t) = \mathcal{O}(t)$, show that
$$\Pi_k(X) = \frac{\vartheta_k(X)}{\log X} + \mathcal{O}\Big(\frac{X}{\log X}\Big).$$
Convince yourself that
$$\vartheta_k(X) \sim kX(\log(\log(X)))^{k-1} \quad (k \ge 2) \tag{6}$$
implies (3) for $k \ge 2$.

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3. So, we have to verify (6). Prove that

$$k\vartheta_{k+1}(X) = (k+1)\sum_{p\leq X}\vartheta_k\Big(\frac{X}{p}\Big) \quad (k\geq 1)$$

   Set $L_0(X) = 1$ and

$$L_k(X) = \sum_{\substack{(p_1,\ldots,p_k)\\ p_1\ldots p_k\leq X}} \frac{1}{p_1\ldots p_k},$$

   and write

$$f_k(X) = \vartheta_k(X) - kxL_{k-1}(X).$$

   Show that (6) follows from

$$f_k(X) = o\Big(X(\log(\log(X)))^{k-1}\Big) \quad (k\geq 1) \tag{7}$$

   and

$$L_k(X) \sim (\log(\log(X)))^k \quad (k\geq 1). \tag{8}$$

4. Equation (7) can be proven by induction. Verify it for $k = 1$. Use

$$kf_{k+1}(X) = (k+1)\sum_{p\leq X} f_k\Big(\frac{X}{p}\Big). \tag{9}$$

   and the fact that

$$\sum_{p\leq X}\frac{1}{p} = \log(\log(X)) + \mathcal{O}(1) \tag{10}$$

   to prove (7) by induction for all $k\geq 2$.

5. We still have to check (8). Demonstrate that

$$\Bigg(\sum_{p\leq X^{\frac{1}{k}}}\frac{1}{p}\Bigg)^k \leq L_k(X) \leq \Bigg(\sum_{p\leq X}\frac{1}{p}\Bigg)^k$$

   and use this as well as (10) to prove (8).

Challenge problem: Prove (3) directly by induction, similar to the proof in exercise 1.

3. Consider an infinite sequence $(a_n)_{n\in\mathbb{N}}$ of integers such that $a_1 < a_2 < a_3 < \ldots$. Further, suppose that there exists an integer $N_0 > 0$ and a constant $c > 0$ such that for all $N > N_0$, there is an $i \in \mathbb{N}$ such that $N^c < a_i < (N+1)^c - 1$. Show that then there is a real number A such that

$$\lfloor A^{c^n}\rfloor \in \mathcal{A} = \{a_i | i\in\mathbb{N}\}$$

for all $n\in\mathbb{N}$.

To prove this, you may proceed in the following way:

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1. Show that there is a subsequence $(b_m)_{m\in\mathbb{N}}$ of $(a_n)_{n\in\mathbb{N}}$ such that

$$b_n^c < b_{n+1} < (b_n + 1)^c - 1.$$

2. Define the sequences $u_m = b_m^{c^{-m}}$ and $v_m = (b_m + 1)^{c^{-m}}$. Show that $u_m < v_m$, $u_m < u_{m+1}$ and $v_n > v_{n+1}$. Deduce from this that the limit

$$A = \lim_{m\to\infty} u_n$$

exists.

3. Conclude that $\lfloor A^{c^m} \rfloor \in \mathcal{A}$ for all m, by showing that $b_m < A^{c^m} < b_m + 1$.

4. As was mentioned in the lecture, M. N. Huxley [1] has shown that for all $\epsilon > 0$,

$$p_{n+1} - p_n < K p_n^{7/12+\epsilon}$$

where p_n denotes the nth prime number and K is some fixed positive integer. Set $N_0 = K^2$ (you also can do better if you like). Use the result of Huxley to prove that for all $N > N_0$, there exists a prime p such that

$$N^3 < p < (N + 1)^3 - 1.$$

Using this, conclude that there exists a real number A such that $\lfloor A^{3^n} \rfloor$ is prime for all $n \in \mathbb{N}$.

Course website: http://www.math.ethz.ch/education/bachelor/lectures/fs2013/math/primes2/

1. M. N. Huxley, *On the differences between consecutive primes*, Invent. math. **15** (1972), 164–170